ON THE ALGEBRAIC K-THEORY OF WITT VECTORS OF FINITE LENGTH

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ABSTRACT. Let k be a perfect field of characteristic p and let $\mathbb{W}_n(k)$ denote the p-typical Witt vectors of length n. For example, $\mathbb{W}_n(\mathbb{F}_p) = \mathbb{Z}/p^n$. We study the algebraic K-theory of $\mathbb{W}_n(k)$, and prove that $K(\mathbb{W}_n(k))$ satisfies "Galois descent". We also compute the K-groups through a range of degrees, and show that the first p-torsion element in the stable homotopy groups of spheres is detected in $K_{2p-3}(\mathbb{W}_n(k))$ for all $n \geq 2$.

1. INTRODUCTION

Let k be a perfect field of characteristic p. Then the algebraic K-theory of k is well understood, at least after p-completion. Indeed, the p-completed K-theory of k is concentrated in degree 0.

The situation is far more complicated, but still well understood, if we lift to characteristic 0 using the Witt vector construction. For example, Bökstedt and Madsen computed the p-completed algebraic K-theory of the p-adic integers $\mathbb{Z}_p = \mathbb{W}(\mathbb{F}_p)$ in [6], and of $\mathbb{W}(\mathbb{F}_{p^s})$ in [7]. Later Rognes [31, 32, 30] computed the K-theory of the 2-adic integers, and Hesselholt and Madsen [22] computed the K-theory of complete discrete valuation fields with residue field k, at least in odd characteristic.

It follows from the work of Hesselholt and Madsen (*loc. cit.*) that $K(\mathbb{W}(k))$ satisfies Galois descent. By that we mean that if $k \to k'$ is a Galois extension of perfect fields of characteristic p then the canonical map $K(\mathbb{W}(k)) \to K(\mathbb{W}(k'))^{hG}$ to the homotopy fixed points of $K(\mathbb{W}(k'))$ is an equivalence on connective covers after p-completion. (This is one version of the Lichtenbaum-Quillen conjecture). But for $\mathbb{W}_n(k)$ for $n < \infty$ the usual tools from algebraic geometry do not work, because $\mathbb{W}_n(k)$ is not a regular ring.

Despite considerable effort, very little is known about $K(\mathbb{W}_n(k))$. Our first main theorem establishes that $K(\mathbb{W}_n(k))$ satisfies Galois descent.

Theorem A. *Suppose $k \to k'$ is a G-Galois extension of perfect fields of characteristic p. Then the canonical map*

$$K(\mathbb{W}_n(k)) \to K(\mathbb{W}_n(k'))^{hG}$$

is an equivalence on connective covers after p-completion for any $n < \infty$.

If we complete at a prime $l \neq p$, we know that $K(\mathbb{W}_n(k))^\wedge_l \simeq K(k)^\wedge_l$. We also know that Galois descent for $k \to k'$ works after completing at any prime. Moreover, if k is finite then $K_*(\mathbb{W}_n(k))$ is finite in each degree. Hence knowing the l-completion for each l suffices to reconstruct $K(\mathbb{W}_n(k))$. It follows that we have Galois descent for finite length Witt vectors of finite fields, see Corollary 4.15.

While the task of understanding $K(\mathbb{W}_n(k))$ completely appears insurmountable with current technology, we do have some partial results. The first of those is the following.

Theorem B. *Suppose $k = \mathbb{F}_q$ is a finite field with q elements. Then*
$$\frac{|K_{2i-1}(\mathbb{W}_n(k),(p))|}{|K_{2i-2}(\mathbb{W}_n(k),(p))|} = q^{(n-1)i}$$
for all $i \geq 1$.

Combining this with Quillen's calculation of $K(\mathbb{F}_q)$ we get a similar result for non-relative K-theory, see Corollary 4.19.

Together with Theorem 5.1 below this goes a long way towards computing $K_*(\mathbb{W}_n(k))$ up to extensions. We can be more explicit in low degrees, determining the groups up to degree $2p-2$.

Theorem C. *Suppose p is a perfect field of characteristic p. Then for any $n \geq 2$ we have*
$$K_{2i-1}(\mathbb{W}_n(k),(p)) \cong \begin{cases} \mathbb{W}_{(n-1)i}(k) & \text{for } 1 \leq 2i-1 \leq 2p-5 \\ \mathbb{Z}/p \oplus R^{-1}(im(\phi-1)) & \text{for } 2i-1 = 2p-3 \end{cases}$$
and
$$K_{2i}(\mathbb{W}_n(k),(p)) \cong \begin{cases} 0 & \text{for } 2 \leq 2i \leq 2p-4 \\ coker(\phi-1) & \text{for } 2i = 2p-2 \end{cases}$$
Here
$$R : \mathbb{W}_{(n-1)(p-1)}(k) \to k$$
is the iterated restriction map and $\phi : k \to k$ is the absolute Frobenius map on k.

Moreover, the unit map from the sphere spectrum sends the first p-torsion element $\alpha_1 \in \pi_{2p-3}S$ to a generator of $\mathbb{Z}/p \subset K_{2p-3}(\mathbb{W}_n(k),(p))$.

If $k = \mathbb{F}_{p^s}$ then $\mathbb{W}_m(k)$ is additively isomorphic to $(\mathbb{Z}/p^m)^s$ and $R^{-1}(im(\phi-1))$ is additively isomorphic to $\mathbb{Z}/p^{m-1} \oplus (\mathbb{Z}/p^m)^{s-1}$. Moreover, $coker(\phi-1) \cong \mathbb{Z}/p$. This allows us to identify the K-theory of $\mathbb{W}_n(\mathbb{F}_{p^s})$, and in particular the K-theory of $\mathbb{Z}/p^n$, explicitly through the same range of degrees, see Corollary 6.5.

We pause to compare this to known results. The calculation of K_1 and K_2 is classical, and the observation that $K_1(\mathbb{W}_n(k)) \cong \mathbb{W}_n(k)^\times$ behaves differently in characteristic 2 is of course even more classical. Theorem C can be thought of as an extension of that phenomenon to odd degrees. In characteristic 3 this was also observed by Geisser [16], who computed $K_3(\mathbb{W}_2(\mathbb{F}_{3^s}))$ when $(3,s) = 1$ and found the extra $\mathbb{Z}/3$ summand coming from the 3-torsion in $\pi_3 S$.

Evens and Friedlander [15] computed K_3 and K_4 of $\mathbb{Z}/p^2$ for $p \geq 5$, but the most general calculation to date, and the only one we know of that goes beyond degree 4, is due to Brun [11] who computed $K_i(\mathbb{Z}/p^n)$ for $i \leq p-3$.

1.1. **Main proof ideas.** We compute using the cyclotomic trace map [5]
$$trc : K(A) \to \mathrm{TC}(A),$$
which for $A = \mathbb{W}_n(k)$ is an equivalence on connective covers after p-completion [21].

The starting point of our calculation is the topological Hochschild homology of k, which looks like $THH(\mathbb{F}_p)$ "tensored up" to k. We can bootstrap from that to $THH_*(\mathbb{W}_n(k))$ by filtering $\mathbb{W}_n(k)$ by powers of p to get a spectral sequence

starting with $THH_*(k[x]/x^n)$, which is known by work of Hesselholt and Madsen [20], and converging to $THH_*(\mathbb{W}_n(k))$. From this we recover Brun's calculation of $THH_*(\mathbb{Z}/p^n)$ [10].

This filtration of $THH(\mathbb{W}_n(k))$ is S^1-equivariant, and as a result we get a corresponding spectral sequence converging to $\mathrm{TF}_*(\mathbb{W}_n(k))$. The restriction map R does not respect the filtration, so we cannot hope to get such a spectral sequence for TC or for K. Instead R divides the filtration by p, and, expanding on ideas of Brun [11], we obtain a spectral sequence converging to $\mathrm{TC}_*(\mathbb{W}_n(k))$.

The proof of Theorem A goes as follows. We first show that $\mathrm{TF}(\mathbb{W}_n(k))$ satisfies Galois descent. This follows becaouse $\mathrm{TF}(k[x]/x^n)$ satisfies Galois descent, plus a collapsing spectral sequence. Then, because homotopy fixed points commute with homotopy equalizers, the same is true for TC and the statement for K-theory follows by taking connective covers.

The proof of Theorem B uses the spectral sequence for TC discussed above. The necessary input is Hesselholt and Madsen's computation of $\mathrm{TC}_*(k[x]/x^n)$ [20], which implies that for $k = \mathbb{F}_q$ we have $|\mathrm{TC}_{2i-1}(k[x]/x^n)| = q^{(n-1)i}$ and $\mathrm{TC}_{2i-2}(k[x]/x^n) = 0$.

For Theorem C we use an idea due to Brun [11] of comparing with cyclic homology, which is more readily understood, as well as the map from $K_*(\mathbb{W}(k))$. Our method breaks down in high degrees because of the possible existence of differentials "crossing filtration n" in the spectral sequence converging to $\mathrm{TC}_*(\mathbb{W}(k),(p))$. If n is sufficiently large it is possible to push the range of degrees for which we understand the K-theory further, but the analysis quickly becomes unwieldy and we omit it.

1.2. **Conventions.** We fix a perfect field k of characteristic p for some prime p throughout and implicitly complete all spectra at p unless we say otherwise. We write $THH(A)$, $\mathrm{TC}(A)$ and $K(A)$ for the (p-completed) topological Hochschild homology, topological cyclic homology, and algebraic K-theory spectrum of A. If a group G acts on a spectrum X, we write X^G, X_{hG}, X^{hG}, X^{gG} and X^{tG} for the fixed point spectrum, homotopy orbit spectrum, homotopy fixed point spectrum, geometric fixed point spectrum, and Tate spectrum, respectively. We let $V(0)$ denote the mod p Moore spectrum, so $V(0)_*X = \pi_*(X;\mathbb{Z}/p)$.

We write $P(x)$, $P_h(x)$, $E(x)$ and $\Gamma(x)$ for a polynomial, truncated polynomial, exterior, and divided powers algebra, respectively. The ground ring will usually be k.

We write R and F for the restriction and Frobenius map, respectively, either from $THH(A)^{C_{p^m}}$ to $THH(A)^{C_{p^{m-1}}}$ or from $\mathbb{W}_{m+1}(k)$ to $\mathbb{W}_m(k)$, and we write ϕ for the absolute Frobenius map on k and its lift to $\mathbb{W}_n(k)$ or $\mathbb{W}(k)$.

1.3. **Acknowledgements.** This paper would never have been started without Mike Hill, with whom I had extensive discussions about the topological Hochschild homology spectral sequence coming from a filtration of a ring. At the time we did not know that Morten Brun had already constructed such a spectral sequence, and we reproved some of his results and did several sample computations together.

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2. A topological Hochschild homology spectral sequence

In this section we study a spectral sequence

$$E_1^{s,t} = \pi_{s+t} THH(GrA; s) \Longrightarrow \pi_{s+t} THH(A)$$

associated to a filtration of a ring A. The existence of this spectral sequence was first noted by Brun [10], though he only used it in an indirect way in his computation of $THH_*(\mathbb{Z}/p^n)$. We will demonstrate that this spectral sequence is a good tool for computations by simplifying and extending known calculations of THH.

For conventions and standard results about spectral sequences, see [4]. Most of the spectral sequences in this paper will be conditionally convergent. If the spectral sequence satisfies some Mittag-Leffler condition it converges strongly. This is typically easy to verify, in most of our examples it follows because the E_1-term is finite (or has finite length over k) in each bidegree. Because of the large number of spectral sequences appearing we will not discuss convergence in each case.

2.1. A Hochschild homology spectral sequence.

We start with Hochschild homology, which is easier, in order to introduce some key ideas. Recall that for a ring A, the Hochschild homology $HH_*(A)$ is the homology of a chain complex $HC_*(A)$ with $A^{\otimes q+1}$ in degree q and

$$d(a_0 \otimes \ldots \otimes a_q) = \sum_{0 \le i \le q-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_q \\ + (-1)^q a_q a_0 \otimes a_1 \otimes \ldots \otimes a_{q-1}.$$

It can also be described as the homology of the cyclic bar construction $B^{cy}_{\otimes}(A)$. If A is graded, we follow the usual sign rule, multiplying by (-1) whenever we move two things (elements, or operators like d) of odd homological degree past each other.

In much of the paper we will need to use derived tensor products. For example, in the definition of Hochschild homology, if A is not projective as a $\mathbb{Z}$-module we replace A by a levelwise projective differential graded ring. For example, $\mathbb{F}_p$ is replaced by the chain complex $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$ with the obvious multiplication. Note that in the literature this version of Hochschild homology is sometimes called Shukla homology.

An alternative description of $HH_*(A)$ is as the homology of the derived tensor product $A \otimes_{A \otimes A^{op}} A$, or as $Tor_*^{A \otimes A^{op}}(A, A)$. The equivalence between the two definitions follows by replacing one of the A's by the 2-sided bar construction $B(A, A, A)$, which is a cofibrant replacement of A as an A-bimodule. In particular, the homology of $\mathbb{Z}/p \otimes \mathbb{Z}/p^{op}$ is exterior over $\mathbb{Z}/p$ on a class in degree 1, and it follows that

$$HH_*(\mathbb{Z}/p) \cong \Gamma(\mu_0)$$

is a divided powers algebra over $\mathbb{Z}/p$ on a class μ_0 in degree 2.

Now suppose $A = \bigoplus A_i$ is a graded ring. In the examples this grading will usually be independent of the homological grading. Then we get a splitting of the Hochschild homology of A.

Lemma 2.1. *Suppose A is a graded ring. Then the Hochschild homology $HH_*(A)$ of A splits as a direct sum*

$$HH_*(A) \cong \bigoplus_s HH_*(A; s),$$

where $HH_*(A; s)$ is the homology of the subcomplex of $HC_*(A)$ of internal degree s. Here we give $a_0 \otimes \ldots \otimes a_q$ in $HC_q(A)$, with each a_i homogeneous, internal degree $|a_0| + \ldots + |a_q|$.

Proof. This is clear, because the Hochschild differential preserves the internal degree. $\square$

Now suppose A is a complete filtered ring. By this we mean that A comes with a decreasing filtration

$$\ldots \to F^{s+1}A \to F^s A \to \ldots \to F^0 A = A.$$

We assume the filtration is compatible with the multiplicative structure, meaning that the multiplication on A induces maps $F^i A \otimes F^j A \to F^{i+j} A$. Complete means that the canonical map $A \to \lim_s A/F^s A$ is an isomorphism. The canonical example comes from an ideal $I \subset A$. If A is I-complete then $F^s A = I^s A$ defines a complete filtration on A. Let $Gr^i A = F^i A/F^{i+1} A$ and let $GrA = \bigoplus_i Gr^i A$. Then GrA is a graded ring, and we can compute $HH_*(GrA)$ as above.

Next we define a corresponding filtration of $HC_*(A)$. We do this by defining

$$F^s HC_q(A) = \bigcup_{i_0 + \ldots + i_q = s} F^{i_0} A \otimes \ldots \otimes F^{i_q} A.$$

It is clear that the Hochschild differential preserves this filtration, so we have a filtration of $HC_*(A)$ and hence a spectral sequence.

Theorem 2.2. *Suppose A is a complete filtered ring with associated graded GrA. Then there is a conditionally convergent spectral sequence*

$$E_1^{s,t} = HH_{s+t}(GrA; s) \Longrightarrow HH_{s+t}(A).$$

The differential d_r has bidegree $(r, -r-1)$. If A is commutative this is an algebra spectral sequence.

Proof. It is clear that we have a spectral sequence converging to $HH_*(A)$ associated to the above filtration, and the E_1-term is as claimed because

$$F^s HC_q(A)/F^{s+1} HC_q(A) \cong \bigoplus_{i_0 + \ldots + i_q = s} Gr^{i_0} A \otimes \ldots \otimes Gr^{i_q} A.$$

This means that $F^s HC_*(A)/F^{s+1} HC_*(A)$ is isomorphic to $HC_*(GrA; s)$.

If A is commutative we have an induced multiplication on each $HC_q(A)$ which descends to a multiplication on $HH_*(A)$, and we get an induced multiplication

$$F^{s_1} HC_q(A) \otimes F^{s_2} HC_q(A) \to F^{s_1+s_2} HC_q(A).$$

This makes the spectral sequence into an algebra spectral sequence. $\square$

Next we look at some examples to show that this spectral sequence can be used quite effectively. We fix a perfect field k of characteristic p. Then the Hochschild homology of k is a divided powers algebra over k on one generator μ_0.

Example 2.3. *First we consider $\mathbb{W}(k)$ filtered by powers of p. Then the associated graded is $Gr\mathbb{W}(k) \cong k[x]$, and we have a (strongly convergent) spectral sequence*

$$E_1^{s,t} = HH_{s+t}(k[x]; s) \Longrightarrow HH_*(\mathbb{W}(k)).$$

We find that

$$HH_*(k[x]) \cong \Gamma(\mu_0) \otimes P(x) \otimes E(\sigma x),$$

where μ_0 comes from $HH_*(k)$. This is bigraded, with $|\mu_0| = (0, 2)$, $|x| = (1, -1)$ and $|\sigma x| = (1, 0)$.

We have an immediate differential

$$d_1(\gamma_j(\mu_0)) = \gamma_{j-1}(\mu_0)\sigma x,$$

for each $j \geq 1$, leaving

$$E_2^{*,*} = E_\infty^{*,*} = P(x)$$

concentrated in homological degree 0. If we use that there is a comultiplication on $E_1^{*,*}$ with $\psi(\gamma_j(\mu_0)) = \sum_{a+b=j} \gamma_a(\mu_0) \otimes \gamma_b(\mu_0)$ we can say that the d_1-differential is generated by the single differential $d_1(\mu_0) = \sigma x$. Since x represents multiplication by p, this recovers the classical result that $HH_0(\mathbb{W}(k)) = \mathbb{W}(k)$ and $HH_i(\mathbb{W}(k)) = 0$ for $i > 0$.

Example 2.4. *Next we consider $\mathbb{W}_n(k)$ filtered by powers of p. Then the associated graded is $Gr\mathbb{W}_n(k) = k[x]/x^n$. Let*

$$E_0^{*,*} = \Gamma(\mu_0) \otimes P_n(x) \otimes E(\sigma x) \otimes \Gamma(x_n),$$

where the new generator x_n has bidegree $|x_n| = (n, 2-n)$. Now define a differential d_0 on $E_0^{*,*}$, generated multiplicatively by $d_0(\gamma_j(x_n)) = nx^{n-1}\gamma_{j-1}(x_n)\sigma x$ for $k \geq 1$. Then

$$HH_*(k[x]/x^n) \cong H_*(E_0^{*,*}, d_0).$$

If p divides n then $d_0 = 0$, and $E_1^{*,*} = E_0^{*,*}$ with a d_1-differential generated multiplicatively by $d_1(\gamma_j(\mu_0)) = \gamma_{j-1}(\mu_0)\sigma x$ for $j \geq 1$, leaving

$$E_2^{*,*} = E_\infty^{*,*} = P_n(x) \otimes \Gamma(x_n).$$

This is the associated graded of $HH_*(\mathbb{W}_n(k)) \cong \mathbb{W}_n(k) \otimes \Gamma(x_n)$. As above, if we use that there is a comultiplication on $E_1^{*,*}$ with $\psi(\gamma_j(\mu_0)) = \sum_{a+b=j} \gamma_a(\mu_0) \otimes \gamma_b(\mu_0)$ we can say that the d_1-differential is generated by the single differential $d_1(\mu_0) = \sigma x$.

If p does not divide n then the E_1-term is somewhat smaller. We still have a d_1-differential generated by $d_1(\gamma_1(\mu_0)) = \sigma x$, but now the E_2-term is somewhat larger. In this case we also have d_2-differentials

$$d_2(x^{n-1}\gamma_j(\mu_0)) = x_n\gamma_{j-2}(\mu_0)\sigma x$$

for $k \geq 2$. This leaves

$$E_3^{*,*} = E_\infty^{*,*} = P_n(x)\{1\} \oplus \bigoplus_{j \geq 1} \big(k\{x^{n-1}\mu_0\gamma_{j-1}(x_n)\} \oplus P_{n-1}(x)\{x\gamma_j(x_n)\}\big).$$

There is a hidden multiplication by p extension, so again we recover that

$$HH_*(\mathbb{W}_n(k)) \cong \mathbb{W}_n(k) \otimes \Gamma(\tilde{x}_n).$$

Now $\gamma_j(\tilde{x}_n)$ is represented by $x^{n-1}\mu_0\gamma_{j-1}(x_n)$, while $p\gamma_j(\tilde{x}_n)$ is represented by $x\gamma_j(x_n)$.

Remark 2.5. *Note that in the above example the case $p \nmid n$ is more complicated. It is possible to filter away this added complexity, as follows. In the Hochschild chain complex $HC_*(\mathbb{W}_n(k))$, introduce a third grading by giving the class representing $\gamma_j(x_n)$ degree $-j$ with associated graded $\widetilde{Gr}HC_*(\mathbb{W}_n(k))$. Then we get a spectral sequence $(\widetilde{Gr}E_r^{*,*}, \tilde{d}_r)$ converging to $\widetilde{Gr}HH_*(\mathbb{W}_n(k))$. The associated graded $\widetilde{Gr}E_0^{*,*}$ is the ring $E_0^{*,*}$ above, now trigraded. Now $d_0 = 0$, because it increases the filtration. Then we get the same d_1-differential as in the case $p \mid n$, at which point the spectral sequence once again collapses. We now have another spectral sequence*

$$E_1^{*,*} = \mathbb{W}_n(k) \otimes \Gamma(x_n) \Longrightarrow HH_*(\mathbb{W}_n(k)),$$

which collapses at the E_1-term, giving us the desired result without having to compute higher differentials.

In anticipation of the proof of Theorem 5.13 below we also explain how to recover $HH_*(\mathbb{W}(k))$ from $HH_*(\mathbb{W}_n(k))$.

Example 2.6. *Now suppose we filter $\mathbb{W}(k)$ by powers of p^n. Then the associated graded is $\mathbb{W}_n(k)[y]$, so we get a spectral sequence*

$$E_1^{s,t} = HH_{s+t}(\mathbb{W}_n(k)[y]; s) \Longrightarrow HH_{s+t}(\mathbb{W}(k)).$$

Then we find that

$$E_1^{*,*} = \mathbb{W}_n(k) \otimes \Gamma(x_n) \otimes P(y) \otimes E(\sigma y).$$

The differentials are generated multiplicatively by

$$d_1(\gamma_j(x_n)) = \gamma_{j-1}(x_n)\sigma y,$$

leaving

$$E_2^{*,*} = E_\infty^{*,*} = \mathbb{W}_n(k) \otimes P(y).$$

This is concentrated in total degree 0, and is the associated graded of $\mathbb{W}(k)$.

2.2. **Topological Hochschild homology.** For a naive definition of THH we have a wide choice of frameworks with which to work. For example, we could define $THH(A)$ as the geometric realization of a simplicial spectrum with $q \mapsto A^{(q+1)}$, the $(q+1)$-fold smash product of A with itself. But to build $THH(A)$ as a cyclotomic spectrum (see Section 3.1 below for the definition of a cyclotomic spectrum) we need a more sophisticated definition. A variant of this definition goes back to Bökstedt [8], see also [21]. Since this technology is well established, we will be brief. See also [1] for a more modern definition.

Let A be a symmetric ring spectrum in the sense of [23], but with topological spaces instead of simplicial sets. If A is a ring, we can regard A as a symmetric ring spectrum by setting $A(i) = K(A, i)$. For each simplicial degree q and finite-dimensional S^1-representation V contained in some complete S^1-universe $\mathcal{U}$ we can consider the space

$$THH(A)_q(V) = \mathrm{hocolim}_{I^{q+1}} \Omega^{i_0+\ldots+i_q}(A(i_0) \wedge \ldots \wedge A(i_q) \wedge S^V).$$

Here I is the category whose objects are $\underline{n} = \{1, \ldots, n\}$ for $n \geq 0$ and whose morphisms are all injective maps. By varying n we get a prespectrum $THH(A)_q$ for each q, and by varying q we get a simplicial prespectrum. We then define the prespectrum $THH(A)$ as the geometric realization of this simplicial prespectrum. Each $THH(A)(V)$ has two S^1-actions, coming from the geometric realization of

a cyclic object and from S^V, and we use the diagonal action. The genuine S^1-spectrum $THH(A)$ is the spectrification of this prespectrum.

Note that while A is a symmetric ring spectrum, $THH(A)$ is a coordinate-free genuine S^1-spectrum in the sense of [25].

In unpublished work [9], Bökstedt computed $THH(\mathbb{F}_p)$ and $THH(\mathbb{Z})$, and in [21] Hesselholt and Madsen extended the first of these calculations to $THH(k)$ for any perfect field k of characteristic p. They found that

$$\pi_* THH(k) \cong P(\mu_0),$$

a polynomial algebra over k on one variable μ_0 in degree 2. Here μ_0 is represented by $1 \otimes \bar{\tau}_0$ in the Bökstedt spectral sequence, where τ_0 is the mod p Bockstein and $\bar{\tau}_0 = -\tau_0$ is its conjugate. The class μ_0 maps to the class with the same name in $HH_2(k)$.

We will see in Example 2.11 below that

$$\pi_j THH(\mathbb{W}(k)) \cong \begin{cases} \mathbb{W}(k) & \text{if } j = 0 \\ \mathbb{W}_{\nu_p(i)}(k) & \text{if } j = 2i-1 \text{ is odd} \\ 0 & \text{if } j \neq 0 \text{ is even} \end{cases}$$

These spectra are sometimes easier to understand if we use mod p coefficients. Let $V(0)$ denote the mod p Moore spectrum. Then

$$V(0)_* THH(\mathbb{W}(k)) \cong E(\lambda_1) \otimes P(\mu_1),$$

where the ground ring is k and $|\lambda_1| = 2p-1$, $|\mu_1| = 2p$.

We can then recover $THH_*(\mathbb{W}(k))$ by running the Bockstein spectral sequence

$$V(0)_* THH(\mathbb{W}(k))[v_0] \Longrightarrow THH_*(\mathbb{W}(k)).$$

This spectral sequence is generated multiplicatively by the differentials

$$d_{j+1}(\mu_1^{p^j}) = v_0^{j+1} \mu_1^{p^j - 1} \lambda_1$$

for $j \geq 0$. If in addition we use the "Leibniz rule" $d_{j+1}(x^p) = v_0 x^{p-1} d_j(x)$ then the Bockstein spectral sequence is generated by the single differential $d_1(\mu_1) = v_0 \lambda_1$.

Remark 2.7. *The "Leibniz rule" in the Bockstein spectral sequence going from mod p homology to integral homology is discussed in* [26, Proposition 6.8]*; at $p = 2$ there is a correction term for d_2 but otherwise it holds. While we have mod p and integral homotopy instead of homology, a similar result holds. The correction term for d_2 at $p = 2$ is $Q^4(\lambda_1)$, and an explicit computation shows that this is indeed* 0.

Returning to the general theory, suppose A is a graded ring. Then we get a splitting of $THH(A)$ into homogeneous pieces in the same way as for Hochschild homology.

Lemma 2.8. *Suppose A is a graded ring or symmetric ring spectrum. Then*

$$THH(A) \cong \bigvee_s THH(A; s),$$

where $THH(A; s)$ is the geometric realization of the subcomplex $THH(A; s)_\bullet$ of internal degree s.

Proof. Define

$$THH(A;s)_q(V) = \bigvee_{s_0+\ldots+s_q=s} \operatorname{hocolim}_{I^{q+1}} \Omega^{i_0+\ldots+i_q}(Gr^{s_0}A(i_0)\wedge\ldots\wedge Gr^{s_q}A(i_q)\wedge S^V).$$

The face and degeneracy maps respect this splitting, hence we get a corresponding splitting after geometric realization. □

2.3. **A topological Hochschild homology spectral sequence.** Now suppose A is a complete filtered ring or symmetric ring spectrum. We can then define a corresponding filtration on $THH(A)$, by setting

$$F^sTHH(A)_q = \bigcup_{s_0+\ldots+s_q=s} F^{s_0}A \wedge \ldots \wedge F^{s_q}A.$$

Here $F^{s_0}A \wedge \ldots \wedge F^{s_q}A$ denotes the spectrification of the genuine S^1-prespectrum

$$V \mapsto \operatorname{hocolim}_{I^{q+1}} \Omega^{i_0+\ldots+i_q}(F^{s_0}A(i_0) \wedge \ldots \wedge F^{s_q}A(i_q) \wedge S^V).$$

We first note that this filtration is compatible with the face and degeneracy maps, so we can define $F^sTHH(A)$ as the geometric realization of $q \mapsto F^sTHH(A)_q$. Hence we have a filtration of $THH(A)$, and we get the following.

Theorem 2.9 (Brun [10])**.** *Suppose A is a complete filtered ring or symmetric ring spectrum with associated graded GrA. Then there is a conditionally convergent spectral sequence*

$$E_1^{s,t} = THH_{s+t}(GrA;s) \Longrightarrow THH_{s+t}(A).$$

If A is commutative this is an algebra spectral sequence.

Proof. As for Hochschild homology, this follows because

$$F^sTHH(A)_q/F^{s+1}THH(A)_q = \bigvee_{s_0+\ldots+s_q=s} Gr^{s_0}A \wedge \ldots \wedge Gr^{s_q}A.$$

This means that $F^sTHH(A)/F^{s+1}THH(A)$ is isomorphic to $THH(GrA;s)$.

If A is commutative the maps

$$F^{s_1}THH(A)_q \wedge F^{s_2}THH(A)_q \to F^{s_1+s_2}THH(A)_q$$

induce an algebra structure on the spectral sequence. □

Remark 2.10. *To get a multiplication on the spectral sequence it suffices to assume that A is an E_2 ring spectrum. This is related to how $THH(A)$ is an S-algebra as long as A is an E_2 ring spectrum, see* [12]*. We omit the details, as we will not need them.*

2.4. **Example computations.** In this section we use Theorem 2.9 to compute $THH(A)$ in some examples.

Example 2.11. *We start by computing $THH_*(\mathbb{W}(k))$ from $THH_*(k[x])$. We find that*

$$E_1^{*,*} = THH_*(k[x]) \cong P(\mu_0) \otimes P(x) \otimes E(\sigma x),$$

where μ_0 comes from $THH_*(k)$. The only difference from Hochschild homology is that here μ_0 is a polynomial generator rather than a divided powers generator.

We have an immediate differential

$$d_1(\mu_0) = \sigma x,$$

because μ_0 is represented by $1 \otimes \bar{\tau}_0$ where τ_0 is the mod p Bockstein and σx is represented by $1 \otimes x$. Hence

$$E_2^{*,*} = P(\mu_1) \otimes P(x) \otimes E(\lambda_1),$$

where $\mu_1 = \mu_0^p$ and $\lambda_1 = x^{p-1}\sigma x$. Next we use the Leibniz rule to get a differential $d_2(\mu_1) = x\lambda_1$, so

$$E_3^{*,*} = P(\mu_2) \otimes P(x) \otimes E(\lambda_2) \oplus \{x\text{-torsion}\}.$$

In general

$$E_{r+1}^{*,*} = P(\mu_r) \otimes P(x) \otimes E(\lambda_r) \oplus \{x\text{-torsion}\},$$

where $\mu_r = \mu_{r-1}^p$ and $\lambda_r = \mu_{r-1}^{p-1}\lambda_{r-1}$, and we recover $THH_*(\mathbb{W}(k))$. Note that the E_2-term of this spectral sequence is isomorphic to the E_1-term of the Bockstein spectral sequence which computes $THH_*(\mathbb{W}(k))$ from $V(0)_*THH(\mathbb{W}(k))[v_0]$.

Example 2.12. *Next we compute $THH_*(\mathbb{W}_n(k))$, starting from $THH_*(k[x]/x^n)$. As for Hochschild homology, the calculation is easier if $p \mid n$. Let*

$$E_0^{*,*} = P(\mu_0) \otimes P_p(x) \otimes E(\sigma x) \otimes \Gamma(x_n)$$

and define a differential d_0 on E_0 by $d_0(x_n) = nx^{n-1}\sigma x$. Then

$$E_1^{*,*} = THH_*(k[x]/x^n) \cong H_*(E_0^{*,*}, d_0).$$

First suppose $p \mid n$. Then $d_0 = 0$, so $E_1^{*,*} = E_0^{*,*}$ and we get the same differentials

$$d_{k+1}(\mu_0^{p^k}) = x^k \mu_0^{p^k-1}\sigma x$$

as for $\mathbb{W}(k)$, for $0 \le k \le n-1$. Next suppose $p \nmid n$. Then, just as in the computation of $HH_*(\mathbb{W}_n(k))$, this moves the differentials around. This is a bit messy, so we prefer to follow the approach in Remark 2.5. As for Hochschild homology, we introduce another filtration on $THH(\mathbb{W}_n(k))$ so that the associated graded is the ring $E_0^{*,*}$ above, now trigraded. This reduces the case $p \nmid n$ to the case $p \mid n$. This proves the following:

Theorem 2.13. *We have*

$$\begin{aligned} THH_{2i}(\mathbb{W}_n(k)) &\cong \bigoplus_{0 \le j \le i} \mathbb{W}_{\max(\nu_p(j),n)}(k) \\ THH_{2i-1}(\mathbb{W}_n(k)) &\cong \bigoplus_{1 \le j \le i} \mathbb{W}_{\max(\nu_p(j),n)}(k) \end{aligned}$$

for all $i \ge 1$.

This recovers Brun's calculation of $THH_*(\mathbb{Z}/p^n)$ from [10]. We note that the first nonzero odd group is $THH_{2p-1}(\mathbb{W}_n(k)) \cong k$, and that the canonical map $THH(\mathbb{W}(k)) \to THH(\mathbb{W}_n(k))$ maps $THH_{2p-1}(\mathbb{W}(k)) \cong k$ isomorphically onto this k.

Example 2.14. *Once again, in anticipation of the proof of Theorem 5.13, we explain how to recover $THH_*(\mathbb{W}(k))$ from $THH_*(\mathbb{W}_n(k))$. If we filter $\mathbb{W}(k)$ by powers of p^n we get a spectral sequence*

$$E_1^{s,t} = THH_{s+t}(\mathbb{W}_n(k)[y]; s) \Longrightarrow THH_{s+t}(\mathbb{W}(k)).$$

Let

$$E_0^{*,*} = \mathbb{W}_n(k) \otimes P(\mu_0) \otimes E(\sigma x) \otimes \Gamma(x_n) \otimes P(y) \otimes E(\sigma y).$$

If $p \mid n$ we find that $E_1^{*,*} = H_*(E_0, d_0)$ where d_0 is multiplicatively generated by $d_0(\mu_0) = \sigma x$. Note that this is $P(\mu_0^{p^n})$-periodic. We then have a differential $d_1(\gamma_j(x_n)) = \gamma_{j-1}(x_n)\sigma y$, which wipes out $\Gamma(x_n)$ and $E(\sigma y)$. We also have the differentials

$$d_{r+1}(\mu_0^{p^{nr}}) = y^r \mu_0^{p^{nr}-1}\sigma x$$

for $r \geq 0$, and this way we recover $THH_*(\mathbb{W}(k))$ from $THH_*(\mathbb{W}_n(k)[y])$.

If $p \nmid n$ the description of $E_1^{*,*}$ is similar, and we have isomorphic differentials.

Observation 2.15. *We note that in the spectral sequence*

$$E_1^{s,t} = THH_{s+t}(\mathbb{W}_n(k)[y]; s) \Longrightarrow THH_{s+t}(\mathbb{W}(k)),$$

all differentials go from even to odd total degree. This will be important in the proof of Theorem B below.

We include one more example. This next example will not be used in the rest of the paper.

Example 2.16. *Consider the Adams summand ℓ of connective p-local complex K-theory $ku_{(p)}$. We filter this by powers of v_1:*

$$\ldots \to \Sigma^{(n+1)(2p-2)}\ell \to \Sigma^{n(2p-2)}\ell \to \ldots \to \ell.$$

This filtration is multiplicative, and the associated graded is

$$Gr\ell \cong H\mathbb{Z}_{(p)}[v_1],$$

where $|v_1| = 2p-2$.

Now, consider the resulting spectral sequence with mod p coefficients. We find that

$$E_1^{*,*} = V(0)_*THH(\mathbb{Z}_{(p)}[v_1]) \cong E(\lambda_1) \otimes P(\mu_1) \otimes P(v_1) \otimes E(\sigma v_1),$$

and there is an immediate differential $d_1(\mu_1) = \sigma v_1$, leaving us with

$$E_2^{*,*} = P(\mu_2) \otimes E(\lambda_1, \lambda_2) \otimes P(v_1).$$

Here $\mu_2 = \mu_1^p$ and $\lambda_2 = \mu_1^{p-1}\sigma v_1$. This coincides with the E_1-term of the v_1-Bockstein spectral sequence considered in [28].

This spectral sequence is also interesting with integral coefficients. Recall from [3] *that in $THH_*(\ell)$ there is an infinite v_1-tower on λ_1 which becomes increasingly p-divisible. In $THH_{2p-1}(\mathbb{Z}_{(p)}[v_1])$ there is a $\mathbb{Z}/p$ generated by λ_1 and a $\mathbb{Z}_{(p)}$ generated by σv_1, and there is a nontrivial extension $p \cdot \lambda_1 = \sigma v_1$ in $THH_*(\ell)$. Hence the class λ_1 is $1/p$ times a naturally defined class.*

We have not attempted to understand the general behavior of the spectral sequence $THH_*(\mathbb{Z}_{(p)}[v_1]) \Longrightarrow THH_*(\ell)$, though it is interesting that with the two spectral sequences in [3] *we now have three spectral sequences converging to $THH_*(\ell)$.*

3. The trace method

In this section we review the "trace method" for computing algebraic K-theory. Most of the material in this section is known, we include it here for the reader's convenience and for ease of reference. In some instances we have generalized known calculations from $\mathbb{F}_p$ or $\mathbb{Z}_p$ to k or $\mathbb{W}(k)$.

3.1. **Fixed points and geometric fixed points.** Recall that a spectrum in the sense of [14] is indexed on a universe $\mathcal{U} \cong \mathbb{R}^\infty$. This means that a spectrum E is an assignment $V \mapsto E(V)$ for each finite-dimensional $V \subset \mathcal{U}$ together with structure maps $\Sigma^W E(V) \to E(V \oplus W)$ such that the adjoint $E(V) \to \Omega^W E(V \oplus W)$ is a homeomorphism.

Following [25] there are two notions of a G-spectrum. A *naive* G-spectrum is simply a spectrum E with a compatible action of G on each $E(V)$. A *genuine* G-spectrum is one indexed on a complete G-universe, a G-inner-product space $\mathcal{U}$ which contains infinitely many copies of each irreducible G-representation.

Given a genuine G-spectrum E, there are two types of G-fixed point spectra. First, we have the usual fixed point spectrum E^G, which is defined space-wise. For $V \in \mathcal{U}^G \subset \mathcal{U}$ we set

$$E^G(V) = E(V)^G.$$

If we take the H-fixed points for some $H \subset G$ we get a genuine $W(H)$-spectrum in the obvious way. It is important to note that taking fixed points does not commute with spectrification. In particular, if X is a G-space then $(\Sigma^\infty_G X_+)^H$ is very different from $\Sigma^\infty_{W(H)} X^H_+$. Instead, the classical tom Dieck splitting gives a formula for $(\Sigma^\infty_G X_+)^H$.

Second, we have the geometric fixed point spectrum E^{gG} (often denoted $\Phi^G(E)$). Recall that for a family $\mathcal{F}$ of subgroups of G which is closed under subconjugacy, there is a G-space $E\mathcal{F}$ with the property that

$$(E\mathcal{F})^H \simeq \begin{cases} * & \text{if } H \in \mathcal{F} \\ \varnothing & \text{if } H \notin \mathcal{F} \end{cases}$$

Now let $\mathcal{F}$ be the family of all proper subgroups, and define $\widetilde{E\mathcal{F}}$ as the cofiber

$$E\mathcal{F}_+ \to S^0 \to \widetilde{E\mathcal{F}}.$$

Then $E^{gG} = (\widetilde{E\mathcal{F}} \wedge E)^G$.

A second, perhaps less intuitive definition is as follows. If $\mathcal{U} = \bigcup V_i$, let $\mathcal{U}^{gG} = \bigcup V_i^G$. Then E^{gG} is the spectrum indexed on $\mathcal{U}^{gG}$ defined as follows. Given $V \in \mathcal{U}^{gG}$, we have $V = W^H$ for some $W \in \mathcal{U}$, and we set

$$E^{gG}(V) = E(W)^G.$$

If we do this for a subgroup $H \subset G$ we again get a genuine $W(H)$-spectrum. Taking geometric fixed points has the property that if X is a G-space then $(\Sigma^\infty_G X_+)^G \cong \Sigma^\infty X^G_+$. More generally, taking geometric fixed points commutes with spectrification, so we can compute E^{gG} at the prespectrum level if we wish. The advantage of using the second definition is that with the right definition of THH it is easy to check that $THH(A)$ is cyclotomic.

Now let $G = S^1$ and let $H = C_n$. Then if E is a genuine S^1-spectrum then E^{gC_n} is a genuine S^1/C_n-spectrum. There is an obvious isomorphism $\rho_n : S^1 \to S^1/C_n$, and we can use this to change E^{gC_n} back into a genuine S^1-spectrum $\rho_n^* E^{gC_n}$.

Definition 3.1 ([21, Definition 2.2])**.** *A genuine S^1-spectrum E is* cyclotomic *if it comes with compatible weak equivalences*

$$\rho_n^* E^{gC_n} \to E$$

for all $n \geq 2$.

The canonical example of a cyclotomic spectrum is $\Sigma^\infty_{S^1} LX_+$, the equivariant suspension spectrum of a free loop space. In this case

$$(\Sigma^\infty_{S^1} LX_+)^{gC_n} \simeq \Sigma^\infty_{S^1/C_n} (LX)^{C_n}_+,$$

and we see that this is a cyclotomic spectrum because $(LX)^{C_n} \cong LX$.

We also know [8, 21] that $THH(A)$ as defined in Section 2.2 is a cyclotomic spectrum. This should not be surprising, since

$$THH(\Sigma^\infty \Omega X_+) \simeq \Sigma^\infty LX_+.$$

Definition 3.2. *Let A be a ring or symmetric ring spectrum. Then the* TR*-groups of A are the homotopy groups of the spectra*

$$\mathrm{TR}^m(A) = THH(A)^{C_{p^{m-1}}}.$$

These spectra are related by a number of maps, in a way that we now recall. There is a map $F : \mathrm{TR}^{m+1}(A) \to \mathrm{TR}^m(A)$ called Frobenius, which is given by inclusion of fixed points.

Definition 3.3. *Let A be a ring or symmetric ring spectrum. Then* $\mathrm{TF}(A)$ *is defined as*

$$\mathrm{TF}(A) = \mathrm{holim}_F \mathrm{TR}^m(A).$$

The Frobenius has an associated transfer map $V : \mathrm{TR}^m(A) \to \mathrm{TR}^{m+1}(A)$ called the verschiebung. There is a map

$$d : \mathrm{TR}^m_q(A) \to \mathrm{TR}^m_{q+1}(A)$$

defined by multiplying by the fundamental class of S^1.

Finally, there is a restriction map

$$R : \mathrm{TR}^{m+1}(A) \to \mathrm{TR}^m(A),$$

which is defined using the cyclotomic structure on $THH(A)$. To be precise, the map

$$R : \mathrm{TR}^2(A) \to \mathrm{TR}^1(A) = THH(A)$$

of non-equivariant spectra is given by the canonical map from fixed points to geometric fixed points, followed by the equivalence of the geometric fixed points with $THH(A)$. More generally $R : \mathrm{TR}^{m+1}(A) \to \mathrm{TR}^m(A)$ is the $C_{p^{m-1}}$ fixed points of this map. If we beef this up to include (virtual) S^1-representations the map R takes the form

$$R : \Sigma^\alpha \mathrm{TR}^{m+1}(A) \to \Sigma^{\alpha'} \mathrm{TR}^m(A),$$

where $\alpha = [\beta] - [\gamma] \in RO(S^1)$ and $\alpha' = \rho_p^*(\alpha^{C_p})$, see [20, 17].

It is generally hard to understand fixed point spectra directly, and it is sometimes useful to compare the actual fixed point spectrum $\mathrm{TR}^{m+1}(A)$ to the homotopy fixed point spectrum $THH(A)^{hC_{p^m}}$. Let $T = THH(A)$, let $T_{hC_{p^m}}$ denote the homotopy orbit spectrum and let $T^{tC_{p^m}}$ denote the Tate spectrum. Then there is a fundamental diagram [6, Theorem 1.10 and Section 2], as follows.

$$\begin{tikzcd}
T_{hC_{p^m}} \arrow[r, "N"] \arrow[d, "="] & \mathrm{TR}^{m+1}(A) \arrow[r, "R"] \arrow[d, "\Gamma_m"] & \mathrm{TR}^m(A) \arrow[d, "\widehat{\Gamma}_m"] \\
T_{hC_{p^m}} \arrow[r, "N^h"] & T^{hC_{p^m}} \arrow[r, "R^h"] & T^{tC_{p^m}}
\end{tikzcd}$$

If we take the homotopy inverse limit over F we obtain a version of the fundamental diagram featuring S^1.

$$\begin{tikzcd}
\Sigma T_{hS^1} \arrow[r, "N"] \arrow[d, "="] & \mathrm{TF}(A) \arrow[r, "R"] \arrow[d, "\Gamma"] & \mathrm{TF}(A) \arrow[d, "\widehat{\Gamma}"] \\
\Sigma T_{hS^1} \arrow[r, "N^h"] & T^{hS^1} \arrow[r, "R^h"] & T^{tS^1}
\end{tikzcd}$$

Now consider the special case $A = \Sigma^\infty \Omega X_+$. Then $THH(A) = \Sigma^\infty_{S^1} LX_+$, where LX denotes the free loop space on X. The tom Dieck splitting says that

$$(\Sigma^\infty_{S^1} LX_+)^{C_{p^m}} \simeq \bigvee_{0 \le k \le m} (\Sigma^\infty LX_+)_{hC_{p^k}}.$$

In this case the top row in the fundamental diagram splits. In general, the existence of the top row in the fundamental diagram can be thought of as a non-split version of the tom Dieck splitting for general A.

Finally we get to topological cyclic homology.

Definition 3.4. *Let A be a ring or symmetric ring spectrum. The topological cyclic homology* $\mathrm{TC}(A)$ *of A is the homotopy equalizer*

$$\mathrm{TC}(A) \to \mathrm{TF}(A) \overset{R}{\underset{id}{\rightrightarrows}} \mathrm{TF}(A).$$

Alternatively, it can be defined as the homotopy equalizer

$$\mathrm{TC}(A) \to \mathrm{TR}(A) \overset{F}{\underset{id}{\rightrightarrows}} \mathrm{TR}(A),$$

where $\mathrm{TR}(A) = \mathrm{holim}_R \mathrm{TR}^m(A)$*, or as* $\mathrm{TC}(A) = \mathrm{holim}_{R,F} \mathrm{TR}^m(A)$*.*

There is a trace map

$$trc : K(A) \to \mathrm{TC}(A)$$

which is an isomorphism on homotopy groups in degree ≥ 0 after p-completion if A is e.g. a finite $\mathbb{W}(k)$-algebra [27]. These comparison results go through relative TC and relative K-theory.

Given a functor F from rings (or symmetric ring spectra) to spectra and an ideal $I \subset A$, we define $F(A, I)$ as the homotopy fiber

$$F(A, I) \to F(A) \to F(A/I).$$

This defines relative K-theory and TC, and we have a relative trace map

$$trc : K(A, I) \to \mathrm{TC}(A, I).$$

What McCarthy [27] actually shows is that this relative trace map is an equivalence after p-completion when I is nilpotent. (Actually the relative trace map is an equivalence even before p-completing, see [13] for details.)

The calculation of $\mathrm{TC}(k)$ recalled below plus Kratzer's calculation of $K(k)$ [24] provides the base case which we use to conclude that the absolute trace map is an equivalence in non-negative degrees after p-completion for certain rings.

In particular this means that up to p-completion we have

$$K_q(\mathbb{W}_n(k), (p)) \cong \mathrm{TC}_q(\mathbb{W}_n(k), (p))$$

for all q.

In some cases we can use a result of Tsalidis to study $\mathrm{TR}^m(A)$ in terms of the C_{p^m} Tate spectrum.

Theorem 3.5 (Tsalidis, [33])**.** *Let A be a connective symmetric ring spectrum of finite type. Suppose*

$$\widehat{\Gamma}_1 : \pi_q THH(A) \to \pi_q THH(A)^{tC_p}$$

is an isomorphism for $q \geq q_0$. Then

$$\widehat{\Gamma}_m : \mathrm{TR}^m_q(A) \to \pi_q THH(A)^{tC_{p^m}}$$

is an isomorphism for $q \geq q_0$ for all m.

This allows for an induction argument, as follows. Recall [18, 6] that there is a Tate spectral sequence converging to $\pi_* THH(A)^{tC_{p^m}}$, and that we get spectral sequences converging to $\pi_* THH(A)_{hC_{p^m}}$ and $\pi_* THH(A)^{hC_{p^m}}$ by (with a small modification in filtration 0) restricting to the first or second quadrant, respectively. If the conditions of Tsalidis' Theorem hold and we understand $\mathrm{TR}^m_*(A)$, we can often understand the spectral sequence converging to $\pi_* THH(A)^{tC_{p^m}}$ because we know what it converges to in degree $q \geq q_0$. Then restricting this spectral sequence to the second quadrant gives a spectral sequence computing $\pi_* THH(A)^{hC_{p^m}}$, and this determines $\mathrm{TR}^{m+1}_q(A)$ for $q \geq q_0$.

By taking the homotopy inverse limit over F, we can also conclude that the maps $\Gamma : \mathrm{TF}_q(A) \to \pi_q THH(A)^{hS^1}$ and $\widehat{\Gamma} : \mathrm{TF}_q(A) \to \pi_q THH(A)^{tS^1}$ are isomorphisms for $q \geq q_0 + 1$.

3.2. **Topological cyclic homology of k and $\mathbb{W}(k)$.** Many computations rely on the corresponding computations for k, so following [21] we spell this case out first. Recall that $THH_*(k) = P(\mu_0)$ is a polynomial algebra (over the ground ring k) on a degree 2 generator μ_0. Then the Tate spectral sequence looks like

$$\widehat{E}_2^{*,*} = P(\mu_0) \otimes E(u_m) \otimes P(t, t^{-1}) \Longrightarrow \pi_* THH(k)^{tC_{p^m}}.$$

This is bigraded by fiber degree and homological degree, with $|\mu_0| = (2, 0)$, $|u_m| = (0, -1)$ and $|t| = (0, -2)$. The topological degree is the sum of the two degrees. The class $v_0 = t\mu_0$ represents multiplication by p and is a permanent cycle. We have a differential

$$d_{2m+1}(u_m) = t^{m+1}\mu_0^m = tv_0^m,$$

leaving

$$\widehat{E}_{2m+2}^{*,*} = \widehat{E}_\infty^{*,*} = P_m(v_0) \otimes P(t, t^{-1}).$$

This is the associated graded of

$$\pi_* THH(k)^{tC_{p^m}} \cong \mathbb{W}_m(k)[t, t^{-1}].$$

When $m = 1$ the map $\widehat{\Gamma}_1 : THH_*(k) \to \pi_* THH(k)^{tC_p}$ is an isomorphism in non-negative degrees and Tsalidis' Theorem applies.

To compute $\pi_* THH(k)^{hC_{p^m}}$ we restrict the Tate spectral sequence to the second quadrant, and we have

$$E_2^{*,*} = P(\mu_0) \otimes E(u_m) \otimes P(t).$$

We have the same d_{2m+1}-differential, which leaves

$$E_{2m+2}^{*,*} = E_\infty^{*,*} = P_m(v_0)\{t^i \mid i > 0\} \oplus P_{m+1}(v_0)\{\mu_0^j \mid j \geq 0\}.$$

This is the associated graded of

$$\pi_* THH(k)^{hC_{p^m}} \cong \mathbb{W}_m(k)\{t^i \mid i > 0\} \oplus \mathbb{W}_{m+1}(k)\{\mu_0^j \mid j \geq 0\}.$$

To compute $R : \pi_* THH(k)^{C_{p^m}} \to \pi_* THH(k)^{C_{p^{m-1}}}$ we need to be a little bit careful. From [21] we know that we have an isomorphism

$$\rho_m : \pi_0 THH(k)^{C_{p^m}} \to \mathbb{W}_{m+1}(k)$$

which is compatible with the restriction map R.

But if we use the map $\Gamma_m : THH(k)^{C_{p^m}} \to THH(k)^{hC_{p^m}}$ to name elements of $\pi_* THH(k)^{C_{p^m}}$ there is another isomorphism that is more natural. We have $\pi_* THH(k)^{hS^1} \cong \mathbb{W}(k)[\mu_0, \mu_0^{-1}]$, and inclusion of fixed points gives us a map $THH(k)^{hS^1} \to THH(k)^{hC_{p^m}}$ which induces an isomorphism

$$\varphi_m : (\pi_* THH(k)^{hC_{p^m}})[0, \infty) \to \mathbb{W}_{m+1}[\mu_0]$$

for each m which is compatible with the Frobenius F.

Lemma 3.6. *Suppose we use the map $\Gamma_m : \pi_* THH(k)^{C_{p^m}} \to \pi_* THH(k)^{hC_{p^m}}$ and the above isomorphism φ_m to name elements of $\pi_* THH(k)^{C_{p^m}}$. Then*

$$R : \pi_0 THH(k)^{C_{p^m}} \to \pi_0 THH(k)^{C_{p^{m-1}}}$$

is identified with

$$R \circ \phi^{-1} : \mathbb{W}_{m+1}(k) \to \mathbb{W}_m(k)$$

where $\phi^{-1} : \mathbb{W}_{m+1}(k) \to \mathbb{W}_{m+1}(k)$ is the inverse of the lift of Frobenius from k to $\mathbb{W}_{m+1}(k)$.

We will typically use the isomorphisms specified in the above lemma, as they are compatible with all the structure except the restriction map, and we find the following (compare [21, Theorem 5.5]).

Lemma 3.7. *Suppose we use φ_m and φ_{m-1} to identify the source and target with $\mathbb{W}_{m+1}(k)[\mu_0]$ and $\mathbb{W}_m(k)[\mu_0]$, respectively. Then the map*

$$R : \pi_* THH(k)^{C_{p^m}} \to \pi_* THH(k)^{C_{p^{m-1}}}$$

is the ring map determined by $R(x) = R_{\mathrm{Witt}} \circ \phi^{-1}(x)$ for $x \in \mathbb{W}_{m+1}(k)$ and $R(\mu_0) = p\lambda_m \mu_0$ for some unit $\lambda_m \in \mathbb{Z}/p^{m+1}$. Here R_{Witt} is the usual restriction map on Witt vectors.

It follows that

$$\mathrm{TC}_i(k) \cong \begin{cases} \mathbb{Z}_p & \text{if } i = 0 \\ coker(\phi - 1) & \text{if } i = -1 \\ 0 & \text{otherwise} \end{cases}$$

(Here we use that $coker(\phi^{-1} - 1) \cong coker(\phi - 1)$.) The trace map $K_*(k) \to \mathrm{TC}_*(k)$ is, after p-completion, an isomorphism in degree 0 and trivial in degree -1, since $K(A)$ is a connective spectrum for any ring A.

Together with Kratzer's calculation [24, Corollary 5.5] of $K(k)$ this provides the base case where the trace map is an equivalence on non-negative homotopy groups after p-adic completion.

Next we consider what happens with $\mathbb{W}(k)$. As for $\mathrm{TR}^m_*(\mathbb{Z})$, we have only been able to determine $\mathrm{TR}^m_*(\mathbb{W}(k))$ up to extensions, so for now we will use mod p coefficients. It follows from Example 2.11 above that

$$V(0)_* THH(\mathbb{W}(k)) \cong E(\lambda_1) \otimes P(\mu_1).$$

If p is odd then $V(0)$ is a ring spectrum and this is a ring isomorphism. If $p=2$ this has to be interpreted additively only (unless we want to invoke the equivalence $V(0) \wedge THH(\mathbb{W}(k)) \simeq THH(\mathbb{W}(k);k)$, which is not S^1-equivariant), although following Rognes [31, 32] our calculation of $V(0)_*\mathrm{TF}(\mathbb{W}(k))$ will still be true additively. We will omit the necessary details required to make our calculation rigorous at $p=2$.

The Tate spectral sequence looks as follows:

$$\widehat{E}_2 = E(\lambda_1) \otimes P(\mu_1) \otimes E(u_m) \otimes P(t,t^{-1}) \Longrightarrow V(0)_*THH(\mathbb{W}(k))^{tC_{p^m}}.$$

The class $v_1 = t\mu_1$ is a permanent cycle. Let $r(j) = p^j + \ldots + p$ for $j \geq 1$. Then there are differentials

$$d_{2r(j)}(t^i) = t^{p^j+i} v_1^{r(j-1)} \lambda_1$$

when $\nu_p(i) = j-1$ for $1 \leq j \leq m$. Finally there is a differential

$$d_{2r(m)+1}(t^i u_m) = t^{p^m+i} v_1^{r(m-1)+1}$$

for $\nu_p(i) \geq m$, after which the spectral sequence collapses. Considering the case $m=1$, the map $V(0)_*THH(\mathbb{W}(k)) \to V(0)_*THH(\mathbb{W}(k))^{tC_p}$ is given by $\lambda_1 \mapsto \lambda_1$ and $\mu_1 \mapsto t^{-p}$, and we see that Tsalidis' Theorem applies. Passing to the S^1-Tate spectrum leaves us with

$$V(0)_*THH(\mathbb{W}(k))^{tS^1} \cong P(v_1) \otimes E(\lambda_1) \oplus \bigoplus_{j \geq 1} P_{r(j)}(v_1)\{t^i \lambda_1 \mid \nu_p(i) = j\}.$$

Restricting to the second quadrant, we find that $V(0)_*THH(\mathbb{W}(k))^{hS^1}$ consists of several parts. To be precise, we have

$$\begin{aligned}
&(3.8) \qquad V(0)_*THH(\mathbb{W}(k))^{hS^1} \cong && P(v_1) \otimes E(\lambda_1) \\
&(3.9) && \bigoplus_{j \geq 1} P_{r(j)}(v_1)\{t^i \lambda_1 \mid \nu_p(i) = j, i \geq p^{j+1}\} \\
&(3.10) && \bigoplus_{j \geq 0} P_{r(j+1)-dp^j}(v_1)\{t^{dp^j} \lambda_1 \mid 0 < d < p\} \\
&(3.11) && \bigoplus_{j \geq 0} P_{r(j+1)}(v_1)\{\mu_1^i \lambda_1 \mid \nu_p(i) = j, i \geq 1\}
\end{aligned}$$

From this we can read off $V(0)_*\mathrm{TF}(\mathbb{W}(k))$, using Tsalidis' Theorem. First, Equation 3.8 comes from a corresponding v_1-tower in $V(0)_*\mathrm{TF}(\mathbb{W}(k))$. Second, Equation 3.9 is concentrated in degree $\leq -2p+1$ so it does not correspond to anything in $V(0)_*\mathrm{TF}(\mathbb{W}(k))$. Third, Equation 3.10 starts in negative degree but

$$B_{d,j} = P_{(p-d)(p^j+\ldots+1)}(v_1)\{v_1^{d(p^{j-1}+\ldots+1)} t^{dp^j} \lambda_1 \mid 0 < d < p\} \tag{3.12}$$

for $0 < d < p$ is in positive degree and corresponds to classes in $V(0)_*\mathrm{TF}(\mathbb{W}(k))$. Finally, the v_1-towers in Equation 3.11 all come from corresponding v_1-towers in $V(0)_*\mathrm{TF}(\mathbb{W}(k))$.

In a similar way we find that

$$\begin{aligned}
V(0)_*THH(\mathbb{W}(k))^{tS^1}[0,\infty) \quad &\cong \quad P(v_1) \otimes E(\lambda_1) \\
&\bigoplus_{j \geq 1} P_{r(j)}(v_1)\{t^i \lambda_1 \mid \nu_p(i) = j, i < 0\} \\
\bigoplus_{0<d<p} \quad &\bigoplus_{j \geq 1} P_{(p-d)(p^{j-1}+\ldots+1)}(v_1)\{v_1^{d(p^{j-1}+\ldots+1)} t^{dp^j} \lambda_1\}
\end{aligned}$$

We can also read off $R : V(0)_*\mathrm{TF}(\mathbb{W}(k)) \to V(0)_*\mathrm{TF}(\mathbb{W}(k))$ this way. If we use the convention in Lemma 3.6 above we find that R is given by ϕ^{-1} on Equation 3.8, maps $B_{d,k+1}$ onto $B_{d,k}$ for $0 < d < p$, and is zero on Equation 3.11.

It follows that

$$V(0)_*\mathrm{TC}(\mathbb{W}(k)) \cong P(v_1)\big\{\mathbb{F}_p\{1, \lambda_1\} \oplus \mathit{coker}(\phi - 1)\{\partial, \partial\lambda_1\} \oplus k\{t^d\lambda_1 | 0 < d < p\}\big\}$$

and

$$V(0)_*K(\mathbb{W}(k)) \cong P(v_1)\big\{\mathbb{F}_p\{1, \lambda_1\} \oplus \mathit{coker}(\phi - 1)\{\partial v_1, \partial\lambda_1\} \oplus k\{t^d\lambda_1 | 0 < d < p\}\big\}$$

Here $|\partial| = -1$, and $v_1^{i-1}t^d\lambda_1$ is represented by

$$\prod_{i \leq (p-d)(p^j+\ldots+1)} v_1^{i-1+d(p^{j-1}+\ldots+1)} t^{dp^j}\lambda_1. \tag{3.13}$$

From this one can determine the homotopy type of $K(\mathbb{W}(k))$, see [7] or [22]. When $p = 2$ the above calculations are still valid when interpreted as $P(v_1^4)$-modules, although determining the homotopy type of the algebraic K-theory spectrum is more complicated.

3.3. **Topological cyclic homology of $k[x]/x^n$.** We also need the computation of $\mathrm{TC}_*(k[x]/x^n)$ from [20]. Suppose Π is a pointed monoid, and let $k(\Pi)$ denote the pointed monoid algebra. Then $THH(k(\Pi)) \simeq THH(k) \wedge B^{cy}_\wedge(\Pi)$, and this is an equivalence of S^1-equivariant spectra. In particular, let $\Pi_n = \{0, 1, x, \ldots, x^{n-1}\}$ so that $k(\Pi_n) = k[x]/x^n$. Then it is clear that $B^{cy}_\wedge(\Pi_n)$ splits as a wedge of homogeneous summands, using the degree in x, and Hesselholt and Madsen found the following.

Theorem 3.14 (Hesselholt-Madsen [20])**.** *The cyclic bar construction $B^{cy}(\Pi_n)$ splits, S^1-equivariantly, as*

$$B^{cy}(\Pi_n) \cong \bigvee_{s \geq 0} B^{cy}(\Pi_n, s),$$

where $B^{cy}(\Pi_n, 0) = S^0$,

$$B^{cy}(\Pi_n, s) \simeq S^1(s)_+ \wedge S^{\lambda_d}$$

if n does not divide s and $B^{cy}(\Pi_n; s)$ sits in a cofiber sequence

$$S^1(s/n)_+ \wedge S^{\lambda_d} \xrightarrow{n} S^1(s)_+ \wedge S^{\lambda_d} \to B^{cy}(\Pi_n, s)$$

if n divides s.

Here $d = \lfloor \frac{s}{n} \rfloor$, $\lambda_d = \mathbb{C}(1) \oplus \ldots \oplus \mathbb{C}(d)$, and $S^1(s)$ denotes S^1 as an S^1-space with an accelerated action. Note that if p does not divide n then

$$B^{cy}(\Pi_n)^\wedge_p \simeq S^0 \vee \bigvee_{n \nmid s} B^{cy}(\Pi_n, s)^\wedge_p.$$

Because the splitting is S^1-equivariant it follows that

$$\mathrm{TR}^m(k[x]/x^n) \cong \bigvee_{s \geq 0} \mathrm{TR}^m(k[x]/x^n, s).$$

We consider the cases $n \nmid s$ and $n \mid s$ separately.

First suppose $n \nmid s$ and consider the Tate spectral sequence

$$\widehat{E}_2^{*,*} = P(\mu_0) \otimes E(e_s) \otimes E(u_m) \otimes P(t, t^{-1})[\lambda_d] \Longrightarrow \mathrm{TR}^m_*(k[x]/x^n, s).$$

The behavior of this spectral sequence depends on m and $\nu_p(s)$. Suppose $m \leq \nu_p(s)$. Then we have the same differential $d_{2m+1}(u_m) = tv_0^m$ as for k, leaving us with

$$\widehat{E}_{2m+2}^{*,*} = \widehat{E}_{\infty}^{*,*} = P_m(v_0) \otimes E(e_s) \otimes P(t, t^{-1})[\lambda_d].$$

This is the associated graded of

$$\pi_* THH(k[x]/x^n, s)^{tC_{p^m}} \cong \mathbb{W}_m(k) \otimes E(e_s) \otimes P(t, t^{-1})[\lambda_d].$$

Restricting this to the second quadrant gives

$$\pi_* THH(k[x]/x^n, s)^{hC_{p^m}} \cong \mathbb{W}_m(k) \otimes E(e_s)\{t^i \mid i > 0\}[\lambda_d] \bigoplus \mathbb{W}_{m+1}(k) \otimes E(e_s)\{\mu_0^j \mid j \geq 0\}[\lambda_d].$$

Now suppose $m \geq \nu_p(s) + 1$. Then we instead have a differential $d_{2\nu_p(s)+2}(1) = e_s t v_0^{\nu_p(s)}$, which leaves

$$\pi_* THH(k[x]/x^n, s)^{tC_{p^m}} \cong \mathbb{W}_{\nu_p(s)}(k) \otimes E(u_m) \otimes P(t, t^{-1})[\lambda_d].$$

Restricting this to the second quadrant gives

$$\pi_* THH(k[x]/x^n, s)^{hC_{p^m}} \cong \mathbb{W}_{\nu_p(s)}(k) \otimes E(u_m)\{t^i e_s \mid i > 0\}[\lambda_d] \bigoplus \mathbb{W}_{\nu_p(s)+1}(k) \otimes E(u_m)\{\mu_0^j e_s \mid j \geq 0\}[\lambda_d].$$

Now we take the inverse limit over m, using the structure map F, and find that for $s \geq 1$ we have

$$\pi_* THH(k[x]/x^n, s)^{tS^1} \cong \mathbb{W}_{\nu_p(s)}(k) \otimes P(t, t^{-1})\{e_s\}[\lambda_d],$$

which is concentrated in odd topological degree. Similarly,

$$\pi_* THH(k[x]/x^n, s)^{hS^1} \cong \mathbb{W}_{\nu_p(s)}(k)\{t^i e_s \mid i > 0\}[\lambda_d] \bigoplus \mathbb{W}_{\nu_p(s)+1}(k)\{\mu_0^j e_s \mid j \geq 0\}[\lambda_d]$$

is concentrated in odd topological degree. Note that this is a $\mathbb{W}_{\nu_p(s)+1}(k)$ in degree $2i+1$ for $i \geq d$ and a $\mathbb{W}_{\nu_p(s)}(k)$ in degree $2i+1$ for $i < d$.

Now suppose $n \mid s$. If $p \nmid n$ then $THH(k[x]/x^n, s)$ is trivial. If $p \mid n$, write $n = ap^v$ with $p \nmid a$. Then the Tate spectral sequence looks like

$$\widehat{E}_2^{*,*} = P(\mu_0) \otimes E(u_m) \otimes P(t, t^{-1})\{e_s, f_s\}$$

with $|e_s| = 1$ and $|f_s| = 2$. If $m < v$ we have the same d_{2m+1}-differential on u_m as before, and if $m \geq v$ we have a differential

$$d_{2v}(f_s) = v_0^k e_s.$$

This leaves us with

$$\widehat{E}_{2v+1}^{*,*} = \widehat{E}_{\infty}^{*,*} = P_v(v_0) \otimes E(u_m) \otimes P(t, t^{-1})\{e_s\}.$$

This is the associated graded of

$$\pi_* THH(k[x]/x^n, s)^{tC_{p^m}} \cong \mathbb{W}_v(k) \otimes E(u_m) \otimes P(t, t^{-1})\{e_s\}.$$

Restricting this to the second quadrant gives

$$\pi_* THH(k[x]/x^n, s)^{hC_{p^m}} \cong \mathbb{W}_v(k) \otimes E(u_m)\{t^i e_s \mid i > 0\}[\lambda_d] \bigoplus \mathbb{W}_v(k) \otimes E(u_m)\{\mu_0^j e_s \mid j \geq 0\}[\lambda_d].$$

Now we take the inverse limit over F, and find that

$$\pi_* THH(k[x]/x^n, s)^{tS^1} \cong \mathbb{W}_v(k) \otimes P(t, t^{-1})\{e_s\}[\lambda_d],$$

which once again is concentrated in odd topological degree. Restricting to the second quadrant once again leaves

$$\pi_* THH(k[x]/x^n, s)^{hS^1} \cong \mathbb{W}_v(k)\{t^i e_s \mid i > 0\}[\lambda_d] \bigoplus \mathbb{W}_v(k)\{\mu_0^j e_s \mid j \geq 0\}[\lambda_d].$$

This does not, in itself, compute $\mathrm{TF}_*(k[x]/x^n)$, because Tsalidis' Theorem does not apply. But it is possible to compute $\mathrm{TF}_*(k[x]/x^n, s)$ for each s directly, identifying it with $\mathrm{TR}^{\nu_p(s)+1}_{*-\lambda_d-1}(k)$ if $n \nmid s$ and with the cokernel of $V^{\nu_p(n)} : \mathrm{TR}^{\nu_p(s/n)+1}_{*-\lambda_d-1}(k) \to \mathrm{TR}^{\nu_p(s)+1}_{*-\lambda_d-1}(k)$ if $n \mid s$. And we have the following computation, see [20]. (See also [17, 2] in the case $k = \mathbb{F}_p$.)

Theorem 3.15. *Let λ be an actual complex S^1-representation. Then $\mathrm{TR}^m_{*-\lambda}(k)$ is concentrated in even degree. If $i \geq \dim_{\mathbb{C}}(\lambda)$ we have $\mathrm{TR}^m_{2i-\lambda}(k) = \mathbb{W}_m(k)$. If $\dim_{\mathbb{C}}(\lambda^{(j-1)}) > i \geq \dim_{\mathbb{C}}(\lambda^{(j)})$ then $\mathrm{TR}^n_{2i-\lambda}(k) = \mathbb{W}_{m-j}(k)$.*

This is proved using an $RO(S^1)$-graded version of the fundamental diagram. For any virtual S^1-representation α we have a fundamental diagram

$$\begin{tikzcd}
T[\alpha]_{hC_{p^m}} \arrow[r, "N"] \arrow[d, "="] & \mathrm{TR}^{m+1}(A)[\alpha] \arrow[r, "R"] \arrow[d, "\Gamma_m"] & \mathrm{TR}^m(A)[\alpha'] \arrow[d, "\widehat{\Gamma}_m"] \\
T[\alpha]_{hC_{p^m}} \arrow[r, "N^h"] & T[\alpha]^{hC_{p^m}} \arrow[r, "R^h"] & T[\alpha]^{tC_{p^m}}
\end{tikzcd}$$

This diagram can also be used to compute $R : \mathrm{TR}^{m+1}_{*-\lambda}(k) \to \mathrm{TR}^m_{*-\lambda'}(k)$.

We use Theorem 3.14 above and find (compare [21, Section 8.2]) that if $n \nmid s$ then

$$\begin{aligned}
\mathrm{TF}(A[x]/x^n; s) &\simeq (S^1(s)_+ \wedge S^{\lambda_d} \wedge THH(A))^{S^1} \\
&\simeq \Sigma F(S^1(s)_+, THH(A) \wedge S^{\lambda_d})^{S^1} \simeq \Sigma(THH(A) \wedge S^{\lambda_d})^{C_s}
\end{aligned}$$

up to p-completion. Similarly, if $n \mid s$ then $\mathrm{TF}(A[x]/x^n; s)$ sits in a cofibration sequence

$$\Sigma(THH(A) \wedge S^{\lambda_d})^{C_{s/n}} \xrightarrow{V_n} \Sigma(THH(A) \wedge S^{\lambda_d})^{C_s} \to \mathrm{TF}(A[x]/x^n; s).$$

Hence

$$\mathrm{TF}_*(k[x]/x^n, s) \cong \mathrm{TR}^{\nu_p(s)+1}_{*-1-\lambda_d}(k)$$

when $n \nmid s$ and similarly for the case $n \mid s$. This is what Hesselholt and Madsen used to compute $K_*(k[x]/x^n)$.

With this we can describe the maps $\Gamma : \mathrm{TF}_*(k[x]/x^n) \to THH(k[x]/x^n)^{hS^1}$ and $\widehat{\Gamma} : \mathrm{TF}_*(k[x]/x^n) \to THH(k[x]/x^n)^{tS^1}$. The map Γ sends $\mathrm{TF}(k[x]x/^n, s)$ to $THH(k[x]/x^n, s)^{hS^1}$ and is given as follows.

Theorem 3.16. *In degree $2i+1$ for $i \geq d$ the map*

$$\Gamma : \mathrm{TF}_{2i+1}(k[x]/x^n; s) \to \pi_{2i+1} THH(k[x]/x^n; s)^{hS^1}$$

is an isomorphism. In degree $2i+1$ for $i < d$ the map

$$\Gamma : \mathrm{TF}_{2i+1}(k[x]/x^n; s) \to \pi_{2i+1} THH(k[x]/x^n; s)^{hS^1}$$

is injective.

We have a similar description of the map $\widehat{\Gamma}$. In this case $\widehat{\Gamma}$ sends $\mathrm{TF}(k[x]/x^n, s)$ to $THH(k[x]/x^n, ps)$.

Theorem 3.17. *In degree $2i+1$ for $i \geq d$ the map*

$$\widehat{\Gamma} : \mathrm{TF}_{2i+1}(k[x]/x^n; s) \to \pi_{2i+1} THH(k[x]/x^n; ps)^{tS^1}$$

is an isomorphism. In degree $2i+1$ for $i < d$ the map

$$\widehat{\Gamma} : \mathrm{TF}_{2i+1}(k[x]/x^n; s) \to \pi_{2i+1} THH(k[x]/x^n; ps)^{tS^1}$$

is injective.

From this we can read off the action of

$$R : \mathrm{TF}_{2i+1}(k[x]/x^n; s) \to \mathrm{TF}_{2i+1}(k[x]/x^n; s/p).$$

Theorem 3.18. *Suppose $\nu_p(s) \geq 1$. In degree $2i+1$ for $i \geq d$ the map*

$$R : \mathrm{TF}_{2i+1}(k[x]/x^n; s) \to \mathrm{TF}_{2i+1}(k[x]/x^n; s/p)$$

is multiplication by p^{i-d}. In degree $2i+1$ for $i < d$ the map R is an isomorphism.

In particular this means that there is a stable range. If s is sufficiently large compared to i then

$$R : \mathrm{TF}_{2i+1}(k[x]/x^n; s) \to \mathrm{TF}_{2i+1}(k[x]/x^n; s/p)$$

is an isomorphism. Here sufficiently large means $i < d$.

4. More spectral sequences

In this section we construct analogues of the spectral sequence from Theorem 2.9 for TR^m and TF, and we note that we have relative versions of all of these spectral sequences. The filtrations necessary to construct these spectral sequences were described by Brun [11], though he only wrote down the filtrations, not the spectral sequences.

We also describe a filtration of TC which comes about in a slightly more complicated way. The restriction map R does not preserve the filtration; it sends $F^s\mathrm{TF}(A)$ to $F^{\lceil s/p \rceil}\mathrm{TF}(A)$, and following Brun [11] once more we define $F^s\mathrm{TC}(A)$ as the homotopy equalizer

$$F^s\mathrm{TC}(A) \to F^s\mathrm{TF}(A) \underset{I}{\overset{R}{\rightrightarrows}} F^{\lceil s/p \rceil}\mathrm{TF}(A),$$

where I is the obvious inclusion.

It is also worth noting that there is a similar filtration of $\Sigma THH(A)_{hS^1}$, with

$$F^s\Sigma THH(A)_{hS^1} \to F^s\mathrm{TF}(A) \xrightarrow{R} F^{\lceil s/p \rceil}\mathrm{TF}(A).$$

Comparing the two spectral sequences will be key to proving Theorem C.

4.1. **Relative THH.** We first note that there is an obvious relative version of the spectral sequence in Theorem 2.9. If A is a complete filtered ring, let $I = F^1 A \subset A$. Then I is an ideal, and the degree 0 part of the associated graded of $THH(A)$ is $THH(A/I)$. Hence the homotopy fiber of $THH(A) \to THH(A/I)$ is $F^1 THH(A)$, and we get a spectral sequence converging to $\pi_* THH(A, I)$ simply by removing the filtration 0 part of the spectral sequence converging to $\pi_* THH(A)$. We state this as a corollary to Theorem 2.9.

Corollary 4.1. *Suppose A is a complete filtered ring or symmetric ring spectrum with associated graded GrA, and let $I = F^1 A \subset A$. Then there is a spectral sequence*

$$E_1^{s,t} = \begin{cases} \pi_{s+t} THH(GrA; s) & \text{if } s \geq 1 \\ 0 & \text{if } s = 0 \end{cases} \Longrightarrow THH_{s+t}(A, I)$$

We analyze the effect of removing filtration 0 in some examples.

Example 4.2. *Consider $THH(\mathbb{W}(k), (p))$ with $\mathbb{W}(k)$ filtered by powers of p. Then we have a spectral sequence*

$$E_1^{*,*} = ker\big(P(\mu_0) \otimes P(x) \otimes E(\sigma x) \to P(\mu_0)\big) \Longrightarrow THH_*(\mathbb{W}(k), (p)).$$

We have essentially the same differentials as before, now with

$$d_{k+1}(x \mu_0^{p^k}) = x^{k+1} \mu_0^{p^k - 1} \sigma x,$$

and this tells us the following.

Theorem 4.3. *We have*

$$THH_q(\mathbb{W}(k), (p)) \cong \begin{cases} p\mathbb{W}(k) & \text{if } q = 0 \\ \mathbb{W}_{\nu_p(i)+1}(k) & \text{if } q = 2i - 1 \text{ is odd} \\ 0 & \text{if } q \geq 2 \text{ is even} \end{cases}$$

In particular the long exact sequence coming from the fiber sequence defining $THH(\mathbb{W}(k), (p))$ degenerates into short exact sequences

$$0 \to THH_{2i}(k) \cong k \to THH_{2i-1}(\mathbb{W}(k), (p)) \cong \mathbb{W}_{\nu_p(i)+1}(k)$$
$$\to THH_{2i-1}(\mathbb{W}(k)) \cong \mathbb{W}_{\nu_p(i)}(k) \to 0.$$

In particular this applies to $THH_*(\mathbb{Z}_p, (p))$. Recall [29] *that the class $\lambda_1 = \mu_0^{p-1} \sigma x \in THH_{2p-1}(\mathbb{Z}_p)$ is in the image of the trace map from $K_{2p-1}(\mathbb{Z}_p, (p))$. Using the relative version of THH we now have classes $\mu_0^i \sigma x \in THH_{2i+1}(\mathbb{Z}_p, (p))$ for all i, and we can ask of any more of these are in the image of the trace map.*

Theorem 4.4. *For $0 \leq i \leq p-1$ the class $\mu_0^i \sigma x \in THH_{2i+1}(\mathbb{Z}_p, (p))$ is in the image of the trace map from $K_{2i+1}(\mathbb{Z}_p, (p))$.*

We prove this theorem right after Theorem 5.12 below.

Example 4.5. *Next we consider $THH(\mathbb{W}_n(k), (p))$. Let*

$$E_0^{*,*} = ker\big(P(\mu_0) \otimes P_p(x) \otimes E(\sigma x) \otimes \Gamma(x_n) \to P(\mu_0)\big)$$

and let d_0 be generated multiplicatively by $d_0(\gamma_k(x_n)) = n x^{n-1} \gamma_{k-1}(x_n)$ for $k \geq 1$. Then we have a spectral sequence

$$E_1^{*,*} = H_*(E_0^{*,*}, d_0) \Longrightarrow THH_*(\mathbb{W}_n(k), (p)).$$

As long as $\nu_p(i) < n$ the following happens. The class μ_0^i was supposed to support a differential, but it is missing, so the target of the differential survives. This gives an extra class in $THH_{2i-1}(\mathbb{W}_n(k),(p))$. If $\nu_p(i) \geq n$ then μ_0^i survives to give a class in $THH_{2i}(\mathbb{W}_n(k))$; running the relative spectral sequence we then get one class less in $THH_{2i}(\mathbb{W}_n(k),(p))$. Hence we find the following (compare Theorem 2.13).

Theorem 4.6. *We have*

$$\begin{aligned} THH_{2i}(\mathbb{W}_n(k),(p)) &\cong \mathbb{W}_{\max(\nu_p(i),n-1)}(k) \oplus \bigoplus_{0\leq j\leq i-1} \mathbb{W}_{\max(\nu_p(j),n)}(k) \\ THH_{2i-1}(\mathbb{W}_n(k),(p)) &\cong \mathbb{W}_{\max(\nu_p(i)+1,n)}(k) \oplus \bigoplus_{1\leq j\leq i-1} \mathbb{W}_{\max(\nu_p(j),n)}(k) \end{aligned}$$

4.2. **A spectral sequence for** TR**.** The spectral sequence in Theorem 2.9 comes from an S^1-equivariant filtration on $THH(A)$, so it is reasonable to expect it to induce a filtration on fixed points as well. Once we have this, we get an induced spectral sequence on fixed points as well.

Theorem 4.7. *Suppose A is a complete filtered ring or symmetric ring spectrum with associated graded GrA. Then there is a spectral sequence*

$$E_1^{s,t} = \mathrm{TR}^m_{s+t}(GrA;s) \Longrightarrow \mathrm{TR}^m_{s+t}(A).$$

If A is commutative then this is an algebra spectral sequence.

Proof. We prove the case $m = 2$, the general case is similar. We use the p-fold edgewise subdivision of the Bökstedt model of THH, which is the spectrification of the genuine S^1-prespectrum with V'th space the geometric realization of

$$THH^{[p]}(A;V)_q = \mathrm{hocolim}_{I^{p(q+1)}} \Omega^{i_0+\ldots+i_{p(q+1)-1}} \big(A(i_0) \wedge \ldots \wedge A(i_{p(q+1)-1}) \wedge S^V\big).$$

The advantage of this model is that we have a simplicial action of C_p.

We have a filtration on each $THH^{[p]}(A;V)$ coming from the filtration on each space $A(i)$ in the spectrum A, and this induces a filtration on $THH^{[p]}(A)$ which is equivalent to the filtration on $THH(A)$ considered before. With this model it is clear that taking fixed points preserves the filtration, since the representation spheres S^V are all in filtration 0. □

There is of course a similar spectral sequence converging to the homotopy groups of the relative spectrum.

Corollary 4.8. *Suppose A is a complete filtered ring or symmetric ring spectrum with associated graded GrA and let $I = F^1A \subset A$. Then there is a spectral sequences*

$$E_1^{s,t} = \begin{cases} \mathrm{TR}^m_{s+t}(GrA;s) & \text{if } s \geq 1 \\ 0 & \text{if } s = 0 \end{cases} \Longrightarrow \mathrm{TR}^m_{s+t}(A,I).$$

A description of the E_1-term of the spectral sequence converging to $\mathrm{TR}^m_*(\mathbb{W}_n(k))$ follows from the calculations in [20], recalled in Section 3.3 above. Because we will only need the corresponding spectral sequence for TF we omit the details.

4.3. **A spectral sequence for** TF**.** The Frobenius F is simply the inclusion of fixed points, so it is compatible with the filtration and we can take a homotopy inverse limit to get a spectral sequence converging to $\mathrm{TF}_*(A)$.

Theorem 4.9. *Suppose A is a complete filtered ring or symmetric ring spectrum with associated graded GrA. Then there is a spectral sequence*

$$E_1^{s,t} = \mathrm{TF}_{s+t}(GrA; s) \Longrightarrow \mathrm{TF}_{s+t}(A).$$

As usual there is a relative version.

Corollary 4.10. *Suppose A is a complete filtered ring or symmetric ring spectrum with associated graded GrA and let $I = F^1A \subset A$. Then there is a spectral sequences*

$$E_1^{s,t} = \begin{cases} \mathrm{TF}_{s+t}(GrA; s) & \text{if } s \geq 1 \\ 0 & \text{if } s = 0 \end{cases} \Longrightarrow \mathrm{TF}_{s+t}(A, I).$$

For $A = \mathbb{W}_n(k)$ this E_1-term is studied in [20] as recalled in the previous section, and we find the following.

Proposition 4.11. *Suppose $p \nmid n$. Then the above spectral sequence converging to $\mathrm{TF}_*(\mathbb{W}_n(k), (p))$ has E_1-term*

$$E_1^{s,*} = \mathrm{TR}^{\nu_p(s)+1}_{*-\lambda_d-1}(k) \qquad \text{if } n \nmid s$$

and $E_1^{s,*} = 0$ if $n \mid s$ for $s \geq 1$.

Note that this is concentrated in odd topological degree, and hence this spectral sequence collapses at the E_1-term. In particular, $E_1^{s,*}$ is a $\mathbb{W}_{\nu_p(s)+1}(k)$ in sufficiently high odd total degree.

Proposition 4.12. *Suppose $p \mid n$. Then the above spectral sequence converging to $\mathrm{TF}_*(\mathbb{W}_n(k), (p))$ has E_1-term*

$$E_1^{s,*} = \begin{cases} \mathrm{TR}^{\nu_p(s)+1}_{*-\lambda_d-1}(k) & \text{if } n \nmid s \\ coker\big(\mathrm{TR}^{\nu_p(s/n)+1}_{*-\lambda_d-1}(k) \xrightarrow{V^{\nu_p(n)}} \mathrm{TR}^{\nu_p(s)+1}_{*-\lambda_d-1}(k)\big) & \text{if } n \mid s \end{cases}$$

for $s \geq 1$

In the case $n \mid s$ the cokernel is isomorphic to $\mathbb{W}_{\nu_p(n)}(k)$ in sufficiently high odd total degree, and again we see that the E_1-term is concentrated in odd topological degree.

Corollary 4.13. *The spectral sequence converging to $\mathrm{TF}_*(\mathbb{W}_n(k), (p))$ collapses at the E_1-term.*

We compare this to $\mathbb{W}(k)$, for which we find the following.

Corollary 4.14. *The spectral sequence converging to $\mathrm{TF}_*(\mathbb{W}(k), (p))$ has E_1-term*

$$E_1^{s,*} = \mathrm{TR}^{\nu_p(s)+1}_{*-1}(k)$$

for $s \geq 1$. This spectral sequence also collapses at the E_1-term.

We can now prove Theorem A.

Proof of Theorem A. Suppose $k \to k'$ is a G-Galois extension of perfect fields of characteristic p. Then it follows from Corollary 4.13 and 4.14 that $\mathrm{TF}_*(\mathbb{W}_n(k')) \cong \mathrm{TF}_*(\mathbb{W}_n(k)) \otimes_{\mathbb{W}(k)} \mathbb{W}(k')$ with the induced G-action. Hence the homotopy fixed point spectral sequence

$$H^*(G; \mathrm{TF}_*(\mathbb{W}_n(k'))) \Longrightarrow \pi_*(\mathrm{TF}(\mathbb{W}_n(k'))^{hG})$$

collapses at the E_2-term, and it follows that the canonical map

$$\mathrm{TF}(\mathbb{W}_n(k)) \to \mathrm{TF}(\mathbb{W}_n(k'))^{hG}$$

is an equivalence.

The maps R and 1 are G-equivariant, and homotopy equalizers commute with homotopy fixed points. Hence the canonical map

$$\mathrm{TC}(\mathbb{W}_n(k)) \to \mathrm{TC}(\mathbb{W}_n(k'))^{hG}$$

is an equivalence as well. The statement of the theorem follows by taking connective covers. □

Now suppose $k \to k'$ is a G-Galois extension of finite fields. Then $K_*(\mathbb{W}_n(k))$ is finite in each degree, and because we have Galois descent after completing at p or completing at $l \neq p$, we find the following.

Corollary 4.15. *Suppose $k \to k'$ is a G-Galois extension of finite fields of characteristic p. Then the canonical map*

$$K(\mathbb{W}_n(k)) \to K(\mathbb{W}_n(k'))^{hG}$$

is an equivalence on connective covers for any $n < \infty$, no completion necessary.

4.4. **A spectral sequence for** TC. For a free loop space LX, we have $(LX)^{C_p} \cong LX$. Given some additive way ℓ to measure the length of a loop, suppose we have $\gamma \in (LX)^{C_p}$. Then $R(\gamma)$ identifies γ, which traverses a loop p times, with the loop traversed just once. Hence we have $\ell(R(\gamma)) = \frac{\ell(\gamma)}{p}$. This works in our situation as well.

Theorem 4.16. *Suppose A is a complete filtered ring or symmetric ring spectrum. Then $R : \mathrm{TR}^{m+1}(A) \to \mathrm{TR}^m(A)$ sends $F^s\mathrm{TR}^{m+1}(A)$ to $F^{\lceil s/p \rceil}\mathrm{TR}^m(A)$ and $R : \mathrm{TF}(A) \to \mathrm{TF}(A)$ sends $F^s\mathrm{TF}(A) \to F^{\lceil s/p \rceil}\mathrm{TF}(A)$.*

Proof. We prove the case $m = 1$, the general case is similar. We use the p-fold edgewise subdivision model of THH considered in the proof of Theorem 4.7 above. Fixed points by the action of C_p are taken spacewise, and a fixed point of a term in the colimit defining $THH^{[p]}(A; V)_q$ looks like

$$(a_0 \wedge \ldots \wedge a_q)^{\wedge p} \wedge v$$

where $v \in (S^V)^{C_p}$. Now, if a_i is homogeneous of filtration $|a_i|$, this is in filtration degree $p(|a_0| + \ldots + |a_q|)$. Applying R replaces this by $(a_0 \wedge \ldots \wedge a_q) \wedge v$, which has filtration degree $|a_0| + \ldots + |a_q|$. □

With this we can make the following definition, compare [11, Section 5].

Definition 4.17. *Suppose A is a complete filtered ring or symmetric ring spectrum. Let $F^s\mathrm{TC}(A)$ denote the homotopy equalizer*

$$F^s\mathrm{TC}(A) \to F^s\mathrm{TF}(A) \underset{I}{\overset{R}{\rightrightarrows}} F^{\lceil s/p \rceil}\mathrm{TF}(A).$$

This provides a filtration of $\mathrm{TC}(A)$. Let $I = F^1A$, and note that because $F^1\mathrm{TF}(A) = \mathrm{TF}(A, I)$ and $\lceil 1/p \rceil = 1$, it follows that $F^1\mathrm{TC}(A) = \mathrm{TC}(A, I)$.

Since we have a filtration we get a spectral sequence, which looks as follows:

Theorem 4.18. *Suppose A is a complete filtered ring or symmetric ring spectrum. Then there is a spectral sequence with*

$$E_1^{s,t} = \ker\big(\mathrm{TF}_{s+t}(GrA;s) \xrightarrow{R} \mathrm{TF}_{s+t}(GrA;s/p)\big) \oplus \mathrm{coker}\big(\mathrm{TF}_{s+t+1}(GrA;s) \xrightarrow{R} \mathrm{TF}_{s+t+1}(GrA;s/p)\big)$$

for $s \geq 1$ and $E_1^{0,t} = \pi_t \mathrm{TC}(A/I)$, converging to $\mathrm{TC}_{s+t}(A)$.

Here $\mathrm{TF}(GrA;s/p) = *$ if p does not divide s. As usual there is a relative version, obtained by removing filtration $s = 0$. When $A/I = k$ the distinction is not important, as $\mathrm{TC}_*(k) = 0$ for $* > 0$.

Proof. It is clear that there is a spectral sequence associated to the filtration, and we can compute the filtration quotients using the following diagram:

$$\begin{tikzcd}
F^{s+1}\mathrm{TC}(A) \arrow[r] \arrow[d] & F^{s+1}\mathrm{TF}(A) \arrow[r, "R-I"] \arrow[d] & F^{\lceil (k+1)/p \rceil}\mathrm{TF}(A) \arrow[d] \\
F^{k}\mathrm{TC}(A) \arrow[r] \arrow[d] & F^{k}\mathrm{TF}(A) \arrow[r, "R-I"] \arrow[d] & F^{\lceil k/p \rceil}\mathrm{TF}(A) \arrow[d] \\
Gr^{k}\mathrm{TC}(A) \arrow[r] & Gr^{k}\mathrm{TF}(A) \arrow[r, "R"] & Gr^{k/p}\mathrm{TF}(A)
\end{tikzcd}$$

□

We can now prove the second main result from the introduction.

Proof of Theorem B. We claim that for $A = \mathbb{W}_n(k)$ filtered by powers of p the spectral sequence in Theorem 4.18 above has the property that all nontrivial differentials go from odd to even total degree. We can be more explicit about the differentials. Given some $x \in E_1^{s,2i-1-s}$ it is represented in $\pi_{2i-1} Gr^s \mathrm{TF}(\mathbb{W}_n(k))$. Since the spectral sequence for TF collapses in this case, it lifts to $\tilde{x} \in \pi_{2i-1} F^s \mathrm{TF}(\mathbb{W}_n(k))$. If $R(\tilde{x}) \in \pi_{2i-1} F^{\lceil s/p \rceil} \mathrm{TF}(\mathbb{W}_n(k))$ is nonzero for all possible lifts $\tilde{x}$, this represents a differential.

On the other hand, a class $y \in E_1^{s,2i-2-s}$ is represented in $\pi_{2i-1} Gr^{s/p} \mathrm{TF}(\mathbb{W}_n(k))$ and y is a permanent cycle by construction of the spectral sequence.

Now fix i and pick N such that

$$R : \mathrm{TF}_{2i-1}(k[x]/x^n; s) \to \mathrm{TF}_{2i-1}(k[x]/x^n; s/p)$$

is an isomorphism for $s \geq N$. To be particular we can choose $N = ni + 1$

Then $E_1^{s,2i-1-s} = E_1^{s,2i-2-s} = 0$ for $s \geq N$ and if $k = \mathbb{F}_q$ a counting argument shows that

$$\frac{|(E_1^{*,*})_{2i-1}|}{|(E_1^{*,*})_{2i-2}|} = \frac{|\bigoplus_{1 \leq s < N} \mathrm{TF}_{2i-1}(k[x]/x^n; s)|}{|\bigoplus_{1 \leq s < N/p} \mathrm{TF}_{2i-1}(k[x]/x^n; s)|} = |\bigoplus_{N/p \leq s < N} \mathrm{TF}_{2i-1}(k[x]/x^n; s)| = q^{(n-1)i}.$$

By using that $K_i(\mathbb{W}_n(k))$ is finite and that localized away from p it is isomorphic to $K_i(k)$ the result follows. □

If instead we use the non-relative K-theory spectrum $K(\mathbb{W}_n(k))$, we pick up an extra $K(k)$ and we get the following.

Corollary 4.19. *Suppose $k = \mathbb{F}_q$ is a finite field with q elements. Then*

$$\frac{|K_{2i-1}(\mathbb{W}_n(k))|}{|K_{2i-2}(\mathbb{W}_n(k))|} = q^{(n-1)i}(q^i - 1)$$

for all $i \geq 2$.

4.5. **Two spectral sequences for $\Sigma THH(A)_{hS^1}$.** We have the usual homotopy orbit spectral sequence converging to $\pi_* \Sigma THH(A)_{hS^1}$ obtained from the Tate spectral sequence by restricting to the first quadrant, but we also have another spectral sequence computing $\pi_* THH(A)_{hS^1}$.

Definition 4.20. *Suppose A is a complete filtered ring or symmetric ring spectrum. Then we let $F^s \Sigma THH(A)_{hS^1}$ denote the homotopy fiber*

$$F^s \Sigma THH(A)_{hS^1} \to F^s \mathrm{TF}(A) \xrightarrow{R} F^{\lceil s/p \rceil} \mathrm{TF}(A).$$

This provides a filtration of $\Sigma THH(A)_{hS^1}$ which is very similar to that of $\mathrm{TC}(A)$. In fact, the filtration quotients are isomorphic and we get a spectral sequence with isomorphic E_1-term.

Theorem 4.21. *Suppose A is a complete filtered ring or symmetric ring spectrum. Then there is a spectral sequence with*

$$\begin{aligned} E_1^{s,t} = \ker\big(&\mathrm{TF}_{s+t}(GrA; s) \xrightarrow{R} \mathrm{TF}_{s+t}(GrA; s/p)\big) \\ &\oplus \operatorname{coker}\big(\mathrm{TF}_{s+t+1}(GrA; s) \xrightarrow{R} \mathrm{TF}_{s+t+1}(GrA; s/p)\big) \end{aligned}$$

for $s \geq 1$ and $E_1^{0,t} = \pi_t \Sigma THH(A/I)_{hS^1}$, converging to $\pi_{s+1} \Sigma THH(A)_{hS^1}$.

The spectral sequences in Theorem 4.18 and Theorem 4.21 not only have isomorphic E_1-terms, the "short" differentials are also isomorphic. By a short differential we mean one which multiplies the filtration by a factor of less than p. This happens because the inclusion map $I : F^s \mathrm{TF}(A) \to F^{\lceil s/p \rceil} \mathrm{TF}(A)$ multiplies the filtration by a factor of p. This phenomenon is closely related to the following result.

Theorem 4.22 (Brun [11, Lemma 5.3])**.** *Suppose $s < t \leq ps$. Then*

$$F^s \mathrm{TC}(A)/F^t \mathrm{TC}(A) \simeq F^s \Sigma THH(A)_{hS^1}/F^t \Sigma THH(A)_{hS^1}.$$

This is especially useful because we can compute $\pi_* THH(\mathbb{W}_n(k))_{hS^1}$ through a range of degrees (compare [11, Proposition 6.4 and 7.2]).

Proposition 4.23. *For $2i \leq 2p - 2$ we have*

$$\pi_{2i} THH(\mathbb{W}_n(k))_{hS^1} \cong \mathbb{W}_{n(i+1)}(k)$$

and for $2i - 1 \leq 2p - 3$ we have

$$\pi_{2i-1} THH((\mathbb{W}_n(k))_{hS^1} = 0.$$

Proof. Recall that the homotopy orbit spectral sequence looks like

$$\pi_* THH(A)[t^{-1}] \Longrightarrow \pi_* THH(A)_{hS^1},$$

and recall that in through degree $2p-2$ we have $\pi_{2i}THH(\mathbb{W}_n(k)) \cong \mathbb{W}_n(k)$ and $\pi_{2i-1}THH(\mathbb{W}_n(k)) = 0$. Hence the homotopy orbit spectral sequence collapses at the E_2-term through the range of degrees we consider. This shows that $\pi_* THH(\mathbb{W}_n(k))_{hS^1}$ has the required length over k.

To show that the extensions are maximally nontrivial, we consider the corresponding homotopy orbit spectral sequence with mod p coefficients:

$$V(0)_* THH(\mathbb{W}_n(k))[t^{-1}] \Longrightarrow V(0)_* THH(\mathbb{W}_n(k))_{hS^1}.$$

Let β_n denote the element in $V(0)_1 THH(\mathbb{W}_n(k))$ which is coming from $p^{n-1} \in THH_0(\mathbb{W}_n(k)) \cong \mathbb{W}_n(k)$. Then we have an immediate differential

$$d_2(t^{-1}) = \beta_n$$

and it follows that we have a differential

$$d_2(t^{-i}) = t^{-i+1}\beta_n$$

for all $i \leq p-1$. This implies that $V(0)_{2i} THH(\mathbb{W}_n(k))_{hS^1} \cong k$ for $2i \leq 2p-2$, and it follows that the extensions are maximally nontrivial. $\square$

Corollary 4.24. *For $2i-1 \leq 2p-3$ we have*

$$\pi_{2i-1} F^1 \Sigma THH(\mathbb{W}_n(k))_{hS^1} \cong \mathbb{W}_{(n-1)i}(k)$$

and for $2i \leq 2p-2$ we have $\pi_{2i} F^1 \Sigma THH(\mathbb{W}_n(k))_{hS^1} = 0$.

5. Comparing spectral sequences

The main goal of this section is to prove that the map

$$\mathrm{TF}_*(\mathbb{W}(k), (p)) \to \mathrm{TF}_*(\mathbb{W}_n(k), (p))$$

is surjective. We need this to prove Theorem C in the next section, and we can also use it to prove the following.

Theorem 5.1. *Let k be a perfect field of characteristic p. Then the canonical map*

$$K_*(\mathbb{W}(k)) \to K_*(\mathbb{W}_n(k))$$

is surjective in even degrees.

We prove this at the end of the section. But first we need some general properties of spectral sequences, and the study of "commutative squares" of spectral sequences. Since we were unable to find a reference we provide proofs. It also requires a trick: we filter $\mathbb{W}(k)$ by powers of p^n and study the corresponding spectral sequence.

5.1. Even-to-odd spectral sequences.

In this section we prove two technical results about spectral sequences where all differentials go from even to odd total degree that will be essential later.

Lemma 5.2. *Suppose we have a map*

$$f : \{E_r^{*,*}\} \to \{\bar{E}_r^{*,*}\}$$

of spectral sequences, and suppose that for some r_0 the map $f : E_{r_0}^{*,*} \to \bar{E}_{r_0}^{*,*}$ is injective in even total degree. Suppose also that any nonzero differential d_r in $\bar{E}_r^{*,*}$ for $r \geq r_0$ goes from even to odd total degree. Then $f : E_r^{*,*} \to \bar{E}_r^{*,*}$ is injective in even total degree for all $r \geq r_0$ and any nonzero differentials d_r in $E_r^{*,*}$ for $r \geq r_0$ goes from even to odd total degree.

Proof. Suppose we have a nonzero differential $d_{r_0}(x) = y$ in $E_{r_0}^{*,*}$ going from odd to even total degree. Then we get a differential $d_{r_0}(f(x)) = f(y)$ in $\bar{E}_{r_0}^{*,*}$, which is nonzero because f is injective in even total degree, a contradicion. This shows that any nonzero differentials d_{r_0} in $E_{r_0}^{*,*}$ goes from even to odd total degree.

To show that $E_{r_0+1}^{*,*} \to \bar{E}_{r_0+1}^{*,*}$ is injective in even total degree it suffices to note that we cannot have a nonzero differential on $\bar{x} \in \bar{E}_{r_0}^{*,*}$ if $\bar{x} = f(x)$ and $d_{r_0}(x) = 0$. The result then follows by induction. $\square$

Next we study the following situation. Suppose A is a spectrum with two compatible filtrations, a "horizontal" filtration with associated graded $Gr^h A$ and a "vertical" filtration with associated graded $Gr^v A$. This means that we have a bifiltration $F^{s,t}A$ of A with maps $F^{s,t}A \to F^{s-1,t}A$ and $F^{s,t}A \to F^{s,t-1}A$ such that the two maps $F^{s,t}A \to F^{s-1,t-1}A$ agree. Also suppose A is complete with respect to both of the filtrations. Then we get a "commutative square" of spectral sequences as follows.

$$\begin{tikzcd}
E_1^{*,*,*} = \pi_* BiGrA \arrow[r, Rightarrow, "SS1"] \arrow[d, Rightarrow, "SS3"'] & E_1'^{*,*} = \pi_* Gr^v A \arrow[d, Rightarrow, "SS2"] \\
E_1''^{*,*} = \pi_* Gr^h A \arrow[r, Rightarrow, "SS4"'] & \pi_* A
\end{tikzcd}$$

Lemma 5.3. *In the above situation, suppose that if we go clockwise around the commutative square of spectral sequences all nonzero differentials go from even to odd total degree. Then the same is true if we go counterclockwise around the commutative square.*

Proof. Suppose we are given

$$x_{s,t} \in \pi_* \frac{F^{s,t}A/F^{s,t+1}A}{F^{s+1,t}A/F^{s+1,t+1}A}$$

of odd total degree. Then by assumption $x_{s,t}$ is an infinite cycle in SS1, this says that x lifts to

$$y_{s,t} \in \pi_* F^{s,t}A/F^{s,t+1}A.$$

Now it is possible that $x_{s,t}$ is killed by a differential in SS1; this happens if and only if the image $y_{-\infty,t}$ of $y_{s,t}$ in $F^{-\infty,t}A/F^{-\infty,t+1}A$ is zero.

To avoid $x_{s,t}$ being hit by a differential, restrict SS1 to filtration $\geq s$, i.e., consider the corresponding spectral sequence converging to $\pi_* F^{s,-\infty}A$. Then $x_{s,t}$ survives, and is represented by $y_{s,t}$. Now we get a corresponding restricted version of SS2, and by assumption $y_{s,t}$ is still an infinite cycle.

To spell out why $y_{s,t}$ is necessarily an infinite cycle, suppose we had $d_r(y_{s,t}) = w_{s,t+r}$ for some nonzero $w_{s,t+r} \in \pi_* F^{s,t+r}A/F^{s,t+r+1}A$. That means that $w_{s,t+r}$ pulls back to a class in $\pi_* F^{s,t+r}A$ which maps nontrivially to $F^{s,t+1}A$ but trivially to $F^{s,t}(A)$. This did not rely on our restricting to filtration $\geq s$, so it contradicts the assumption that SS2 does not have any differentials going from odd to even degree.

Hence $y_{s,t}$ lifts to a class $z_{s,t}$ in $F^{s,t}A$. Then $z_{s,t}$ and its image in $F^{s,t}A/F^{s+1,t}A$ provide the required lifts showing that $x_{s,t}$ is indeed an infinite cycle in SS3 and SS4. $\square$

5.2. **The bifiltered Tate spectrum.** Suppose A is a complete filtered ring or symmetric ring spectrum and we want to compute $\pi_* THH(A)^{tS^1}$. Then we have two filtrations of $THH(A)^{tS^1}$, by the filtration coming from A and by the Tate filtration. To be able to compare the spectral sequences more easily, we double the grading coming from GrA. This has the effect of doubling the length of the differentials in that spectral sequence. We then get a commutative square

$$\begin{array}{ccc} E_2^{*,*,*} = \pi_* THH(GrA) \otimes P(t,t^{-1}) & \Longrightarrow & E_2'^{*,*} = \pi_* THH(A) \otimes P(t,t^{-1}) \\ \Downarrow & & \Downarrow \\ E_2''^{*,*} = \pi_* THH(GrA)^{tS^1} & \Longrightarrow & \pi_* THH(A)^{tS^1} \end{array}$$

To spell this out, we have a horizontal spectral sequence

$$\{(E_r^{h\,*,*}, d_r^h)\}_{r\geq 2} \Longrightarrow \pi_* THH(A) \otimes P(t,t^{-1}) \tag{5.4}$$

with $E_2^{h\,*,*} = E_2^{*,*}$. Here we ignore the grading on $E_2^{*,*,*}$ coming from Tate cohomology; it is preserved by all the differentials. Similarly we have a vertical spectral sequence

$$\{(E_r^{v\,*,*}, d_r^v)\}_{r\geq 2} \Longrightarrow \pi_* THH(GrA)^{tS^1} \tag{5.5}$$

with $E_2^{v\,*,*} = E_2^{*,*}$, where this time we ignore the grading on $E_2^{*,*,*}$ coming from the grading on GrA. We also have the classical Tate spectral sequence

$$E_2'^{*,*} = THH_*(A) \otimes P(t,t^{-1}) \Longrightarrow \pi_* THH(A)^{tS^1} \tag{5.6}$$

as well as a spectral sequence

$$E_2''^{*,*} = \pi_* THH(GrA)^{tS^1} \Longrightarrow \pi_* THH(A)^{tS^1}. \tag{5.7}$$

We have a similar commutative square for computing homotopy fixed points or homotopy orbits, with coefficients, or for the corresponding relative spectra.

Example 5.8. *We first consider the commutative square of spectral sequences for $\pi_* THH(\mathbb{W}(k),(p))^{tS^1}$. In this case the commutative square looks as follows.*

(5.9)

$$\begin{array}{ccc} E_2^{*,*,*} = THH_*(k[x],(x)) \otimes P(t,t^{-1}) & \Longrightarrow & E_2'^{*,*} = THH_*(\mathbb{W}(k),(p)) \otimes P(t,t^{-1}) \\ \Downarrow & & \Downarrow \\ E_2''^{*,*} = \pi_* THH(k[x],(x))^{tS^1} & \Longrightarrow & \pi_* THH(\mathbb{W}(k),(p))^{tS^1} \end{array}$$

We know that $THH(k[x],(x))^{tS^1} \cong \bigvee_{s\geq 1} THH(k[x];s)^{tS^1}$. We have

$$THH_*(k[x];s) \cong P(\mu_0)\{x^s, x^{s-1}\sigma x\},$$

and in the left hand side vertical spectral sequence we have

$$d^v_{2\nu_p(s)+2}(x^s) = t v_0^{\nu_p(s)} x^{s-1}\sigma x,$$

where we remember that $v_0 = t\mu_0$. This leaves

$$P_{\nu_p(s)}(v_0) \otimes P(t,t^{-1})\{x^{s-1}\sigma x\}.$$

This follows from $B^{cy}(\Pi_\infty; s) \simeq S^1(s)_+$, but we can also think about it in the following way. We have an immediate differential $d_2(x) = t\sigma x$, and now the rest of

the differentials follow from the Leibniz rule. In this case v_0 represents p, and the Leibniz rule says that $d_{r+2}(a^p) = v_0 a^{p-1} d_r(a)$.

Note that in this case the horizontal and vertical spectral sequences are almost abstractly isomorphic, with the roles of x and v_0 interchanged. The only difference is that we have taken the kernel of the map to $P(\mu_0)$, so these classes are missing but the corresponding copy of $P(x)$ is still present. Hence ${E'_2}^{*,*}$ is slightly larger than ${E''_2}^{*,*}$.

The spectral sequence in Equation 5.7 collapses, because everything is concentrated in odd total degree. Hence we would expect the extra classes in ${E'_2}^{*,*}$ to kill each other off. These consist of $xP(x) \otimes P(t, t^{-1})$ in even total degree and $v_0^{s-1} x^{\nu_p(s)} \sigma x \otimes P(t, t^{-1})$ for $s \geq 1$ in odd total degree.

Theorem 5.10. *In the Tate spectral sequence in Equation 5.6 which converges to $\pi_* THH(\mathbb{W}(k), (p))^{tS^1}$ we have $P(t, t^{-1})$-linear differentials*

$$d_{2s}(p^s) = t v_0^{s-1} p^{\nu_p(s)} \sigma x$$

for each $s \geq 1$.

Proof. This follows by a counting argument, using the homotopy orbit spectrum. The point is that in the diagram

$$\begin{array}{ccc} \pi_* THH(k[x], (x)) \otimes P(t^{-1}) & \Longrightarrow & \pi_* THH(\mathbb{W}(k), (p)) \otimes P(t^{-1}) \\ \Downarrow & & \Downarrow \\ \pi_* THH(k[x], (x))_{hS^1} & \Longrightarrow & \pi_* THH(\mathbb{W}(k), (p))_{hS^1} \end{array}$$

we can read off the length of $\pi_{2i-1} THH(\mathbb{W}(k), (p))_{hS^1}$ for all i by going around counter-clockwise. Going around clockwise the differentials must then be as claimed. $\square$

The upshot of all of this is that the E_∞-term of the Tate spectral sequence in Equation 5.9 is isomorphic to the E_2-term of the absolute Tate spectral sequence

$$\widehat{E}_2^{*,*} = THH_*(\mathbb{W}(k)) \otimes P(t, t^{-1}) \Longrightarrow \pi_* THH(\mathbb{W}(k))^{tS^1}$$

with fiber degree 0 removed.

Example 5.11. *Next we consider $V(0)_* THH(\mathbb{W}(k), (p))^{tS^1}$. We find that except for degree zero the map*

$$V(0)_* THH(\mathbb{W}(k))^{tS^1} \to V(0)_* THH(k)^{tS^1}$$

is trivial. One might wish to argue that we can compute $V(0)_* THH(\mathbb{W}(k), (p))$ by removing fiber degree 0 of the E_2-term of the Tate spectral sequence converging to $V(0)_* THH(\mathbb{W}(k))^{tS^1}$.

While this does compute the correct answer, it is more difficult to justify because $V(0)_* THH(\mathbb{W}(k), (p))$ is not isomorphic to $V(0)_* THH(\mathbb{W}(k))$ in positive degrees, so the spectral sequence

$$E_2^{*,*} = V(0)_* THH(\mathbb{W}(k), (p)) \otimes P(t, t^{-1}) \Longrightarrow V(0)_* THH(\mathbb{W}(k), (p))^{tS^1}$$

looks quite different to the corresponding spectral sequence for $THH(\mathbb{W}(k))$.

Instead we use that in positive degree we have

$$V(0)_* THH(\mathbb{W}(k), (p))^{tS^1} \cong V(0)_* THH(\mathbb{W}(k))^{tS^1} \oplus V(0)_{*+1} THH(k)^{tS^1}.$$

Recall that

$$V(0)_*THH(k)^{tS^1} \cong P(t,t^{-1}).$$

Now we argue as follows. Recall that in $V(0)_*THH(\mathbb{W}(k))^{tS^1}$ *we have truncated* v_1*-towers of the form*

$$P_{r(j)}(v_1)\{t^i\lambda_1\}$$

whenever $\nu_p(i)=j$. *Since the map*

$$V(0)_*THH(\mathbb{W}(k),(p)) \to V(0)_*THH(\mathbb{W}(k))$$

is injective, we have a class $v_1^{r(j)-1}t^i\lambda_1$ *in* $V(0)_*THH(\mathbb{W}(k),(p))$. *We find that* $v_1\cdot(v_1^{r(j-1)-1}t^i\lambda_1)$ *maps to 0 in* $V(0)_*THH(\mathbb{W}(k))^{tS^1}$. *But this class is killed by* t^{i-p^j} *in the spectral sequence converging to* $V(0)_*THH(\mathbb{W}(k))^{tS^1}$, *and this means that it is nonzero and represented by* ∂t^{i-p^j} *in* $V(0)_*THH(\mathbb{W}(k),(p))^{tS^1}$. *Hence we find the following:*

Theorem 5.12. *We have*

$$\begin{aligned} V(0)_*THH(\mathbb{W}(k),(p))^{tS^1}[0,\infty) \;\cong\;& P(v_1)\otimes E(\lambda_1) \\ & \textstyle\bigoplus_{j\geq 0} P_{r(j)+1}(v_1)\{t^i\lambda_1 \mid \nu_p(i)=j, i<0\} \\ \bigoplus_{0<d<p} & \textstyle\bigoplus_{j\geq 0} P_{(p-d)(p^{j-1}+\ldots+1)+1}(v_1)\{v_1^{d(p^{j-1}+\ldots+1)}t^{dp^j}\lambda_1\} \end{aligned}$$

This means that every v_1*-tower is one longer. This will make some of our results just a little bit stronger. In particular we can use it to prove Theorem 4.4.*

Proof of Theorem 4.4. If we compute $V(0)_*THH(\mathbb{Z}_p,(p))^{hS^1}$ as well we also see truncated v_1-towers that are one longer. We find that

$$V(0)_*\mathrm{TC}(\mathbb{Z}_p,(p)) = P(v_1)\otimes E(\lambda_1,\partial)\oplus \bigoplus_{0<d<p} P(v_1)\{t^d\lambda_1\}$$

as before, but now with $v_1^{i-1}t^d\lambda_1$ represented by

$$\prod_{i\leq (p-d)(p^j+\ldots+1)+1} v_1^{i-1+d(p^{j-1}+\ldots+1)}t^{dp^j}\lambda_1,$$

compare Equation 3.13. In particular, this class maps to the class named $t^d\lambda_1$ with one naming convention, and $\mu_0^{p-d-1}\sigma x$ with another naming convention, in $THH_{2p-1-2d}(\mathbb{Z}_p,(p))$. □

5.3. **The map from** $\mathrm{TF}_*(\mathbb{W}(k),(p))$ **to** $\mathrm{TF}_*(\mathbb{W}_n(k),(p))$**.** In this section we study the map $\mathrm{TF}_*(\mathbb{W}(k),(p)) \to \mathrm{TF}_*(\mathbb{W}_n(k),(p))$ and prove Theorem 5.1. In particular, we prove the following.

Theorem 5.13. *The canonical map*

$$\mathrm{TF}_*(\mathbb{W}(k),(p)) \to \mathrm{TF}_*(\mathbb{W}_n(k),(p))$$

is surjective in all degrees.

Proof. Consider filtering $\mathbb{W}(k)$ by powers of p^n. Then consider the following "commutative diagram".

$$\begin{tikzcd}
THH_*(\mathbb{W}_n(k)[y],(p,y)) \otimes P(t) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & THH_*(\mathbb{W}(k),(p)) \otimes P(t) \arrow[d, Rightarrow] \\
\pi_* THH(\mathbb{W}_n(k)[y],(p,y))^{hS^1} \arrow[r, Rightarrow] & \pi_* THH(\mathbb{W}(k),(p))^{hS^1} \\
\mathrm{TF}_*(\mathbb{W}_n(k)[y],(p,y)) \arrow[r, Rightarrow] \arrow[u, "\Gamma"] & \mathrm{TF}_*(\mathbb{W}(k),(p)) \arrow[u, "\Gamma"']
\end{tikzcd}$$

Going clockwise around the top square all differentials go from even to odd total degree (see Observation 2.15), hence by Lemma 5.3 so do the differentials going counterclockwise around the top square.

The left hand side map labeled Γ splits as a wedge of

$$\Gamma^0 : \mathrm{TF}_*(\mathbb{W}_n(k),(p)) \to \pi_* THH(\mathbb{W}_n(k),(p))^{hS^1}$$

and

$$\Gamma^s : \mathrm{TF}_*(\mathbb{W}_n(k)[y];s) \to \pi_* THH(\mathbb{W}_n(k)[y];s)^{hS^1}$$

for $s \geq 1$. It follows from Theorem 3.16 that Γ^0 is injective, and we know that the right hand side map labeled Γ is injective. Since $\mathrm{TF}_*(\mathbb{W}_n(k),(p))$ is concentrated in odd total degree it follows that the image survives the middle horizontal spectral sequence, and hence $\mathrm{TF}_*(\mathbb{W}_n(k),(p))$ survives the bottom horizontal spectral sequence.

The map $\mathrm{TF}_*(\mathbb{W}(k),(p)) \to \mathrm{TF}_*(\mathbb{W}_n(k),(p))$ is obtained from the bottom horizontal spectral sequence by restricting to filtration 0, i.e., to $\mathrm{TF}_*(\mathbb{W}_n(k),(p))$, and the differentials originating from $\mathrm{TF}_*(\mathbb{W}_n(k),(p))$ in the spectral sequence measure the failure of this map to be surjective. Since there are none, the result follows. □

Proof of Theorem 5.1. From Theorem 5.13 we have a short exact sequence

$$0 \to \mathrm{TF}_{2i-1}(\mathbb{W}(k),(p^n)) \to \mathrm{TF}_{2i-1}(\mathbb{W}(k),(p)) \to \mathrm{TF}_{2i-1}(\mathbb{W}_n(k),(p)) \to 0.$$

By considering the kernel and cokernel of $R-1$ we get a 6-term exact sequence

$$\begin{aligned} 0 &\to \mathrm{TC}_{2i-1}(\mathbb{W}(k),(p^n)) \to \mathrm{TC}_{2i-1}(\mathbb{W}(k),(p)) \to \mathrm{TC}_{2i-1}(\mathbb{W}_n(k),(p)) \\ &\to \mathrm{TC}_{2i-2}(\mathbb{W}(k),(p^n)) \to \mathrm{TC}_{2i-2}(\mathbb{W}(k),(p)) \to \mathrm{TC}_{2i-2}(\mathbb{W}_n(k),(p)) \to 0 \end{aligned}$$

and the result follows. □

6. Proof of Theorem C

In this section we prove Theorem C. Given a filtered object X and integers $a < b$, it will be convenient to use the notation $F^{[a,b]}X$ for $F^aX/F^{b+1}X$.

6.1. An isomorphism between filtered pieces of $\mathrm{TC}(\mathbb{W}(k))$ and $\mathrm{TC}(\mathbb{W}_n(k))$.

We prove the following results.

Proposition 6.1. *Let $i \geq 2$. Then the canonical map $\mathrm{TF}(\mathbb{W}(k)) \to \mathrm{TF}(\mathbb{W}_n(k))$ induces an isomorphism*

$$\pi_{2i-1} F^{[1,2n-2+\epsilon]} \mathrm{TF}(\mathbb{W}(k)) \cong \pi_{2i-1} F^{[1,2n-1]} \mathrm{TF}(\mathbb{W}_n(k)).$$

Here

$$F^{[1,2n-2+\epsilon]}\mathrm{TF}(\mathbb{W}(k)) = F^1\mathrm{TF}(\mathbb{W}(k))/(p^{\nu_p(2n-1)}F^{2n-1}\mathrm{TF}(\mathbb{W}(k)) \cup F^{2n}\mathrm{TF}(\mathbb{W}(k))).$$

In particular, if $\nu_p(2n-1) = 0$ then this is just $F^{[1,2n-2]}\mathrm{TF}(\mathbb{W}(k))$.

Proof. Theorem 5.13 above says in particular that the map $\mathrm{TF}_{2i-1}(\mathbb{W}(k),(p)) \to \mathrm{TF}_{2i-1}(\mathbb{W}_n(k),(p))$ is surjective. On the associated graded we find the following. For $1 \leq s \leq n-1$ the map $Gr^s\mathrm{TF}_{2i-1}(\mathbb{W}(k)) \to Gr^s\mathrm{TF}_{2i-1}(\mathbb{W}_n(k))$ is an isomorphism. Then the map $Gr^n\mathrm{TF}_{2i-1}(\mathbb{W}(k)) \to Gr^n\mathrm{TF}_{2i-1}(\mathbb{W}_n(k))$ is surjective with kernel k. For $n+1 \leq s \leq 2n-1$, $Gr^s\mathrm{TF}_{2i-1}(\mathbb{W}(k)) \to Gr^s\mathrm{TF}_{2i-1}(\mathbb{W}_n(k))$ is multiplication by p. See [19, Lemma 5.3] for the case $k = \mathbb{F}_p$, the general case follows by considering the inclusion $\mathbb{F}_p \to k$.

The only way for $\mathrm{TF}_{2i-1}(\mathbb{W}(k),(p)) \to \mathrm{TF}_{2i-1}(\mathbb{W}_n(k),(p))$ to be surjective in this range of filtrations is for the following to happen. For each $n \leq s \leq 2n-2$, $p^{\nu_p(s)}$ times any lift of the generator of $Gr^s\mathrm{TF}_{2i-1}(\mathbb{W}(k))$ to $\mathrm{TF}_{2i-1}(\mathbb{W}(k),(p))$ must map to a lift of the generator of $Gr^{s+1}\mathrm{TF}_{2i-1}(\mathbb{W}_n(k))$ to $\mathrm{TF}_{2i-1}(\mathbb{W}_n(k),(p))$. The result follows. □

Proposition 6.2. *Suppose $p \geq 3$, $i \geq 3$, and that there exists some $2n+1 \leq s_0 \leq 3n-1$ with $p \mid s_0$. Then the canonical map $\mathrm{TF}(\mathbb{W}(k)) \to \mathrm{TF}(\mathbb{W}_n(k))$ induces an isomorphism*

$$\pi_{2i-1}F^{[1,s_0-1+\epsilon]}\mathrm{TF}(\mathbb{W}(k)) \cong \pi_{2i-1}F^{[1,s_0]}\mathrm{TF}(\mathbb{W}_n(k)).$$

Here

$$F^{[1,s_0-1+\epsilon]}\mathrm{TF}(\mathbb{W}(k)) = F^1\mathrm{TF}(\mathbb{W}(k))/(p^{\nu_p(s_0)-1}F^{s_0}\mathrm{TF}(\mathbb{W}(k)) \cup F^{s_0+1}\mathrm{TF}(\mathbb{W}(k))).$$

In particular, if $\nu_p(s_0) = 1$ then this is just $F^{[1,s_0-1]}\mathrm{TF}(\mathbb{W}(k))$.

Proof. The proof is similar to the proof of the previous result, starting from the fact that the map $\pi_{2i-1}Gr^{s_0}\mathrm{TF}(\mathbb{W}(k)) \to \pi_{2i-1}Gr^{s_0}\mathrm{TF}(\mathbb{W}_n(k))$ has kernel $\mathbb{W}_2(k)$. Hence the map $\mathrm{TF}_{2i-1}(\mathbb{W}(k),(p)) \to \mathrm{TF}_{2i-1}(\mathbb{W}_n(k),(p))$ must increase the filtration by 2 in this range. □

Proposition 6.3. *In total degree less than or equal to $2p-3$ the differentials in the spectral sequences converging to $\pi_*\mathrm{TC}(\mathbb{W}_n(k))$ and to $\pi_*\Sigma THH(\mathbb{W}_n(k))_{hS^1}$ are isomorphic.*

Proof. The spectra $\mathrm{TC}(\mathbb{W}_n(k))$ and $\Sigma THH(\mathbb{W}_n(k))_{hS^1}$ are both the homotopy fiber of maps $\mathrm{TF}(\mathbb{W}_n(k)) \to \mathrm{TF}(\mathbb{W}_n(k))$, with the map for $\mathrm{TC}(\mathbb{W}_n(k))$ being $R-I$ and the map for $\Sigma THH(\mathbb{W}_n(k))_{hS^1}$ being R. With our conventions I multiplies the filtration by p, so all differentials which increase the filtration by a factor of less than p will be the same in both cases. Let us call a differential which increases the filtration by a factor of at least p a long differential.

Now suppose there is such a long differential on a class x in filtration j in the spectral sequence converging to $\pi_*\Sigma THH(\mathbb{W}_n(k))_{hS^1}$. Through this range of degrees there are no nontrivial targets in filtration $\geq n(p-1)+1$, so we must have $j < n$.

Now consider the corresponding spectral sequences for $\mathbb{W}_j(k)$. There is a class representing x in filtration j or $j+1$, which must now survive to E_∞ in the spectral sequence converging to $\pi_*\Sigma THH(\mathbb{W}_j(k))_{hS^1}$. But this leads to a contradiction, because x does not represent a multiple of the generator of $\pi_{2i-1}\Sigma THH(\mathbb{W}_j(k))_{hS^1}$. □

6.2. **Proof of Theorem C.** Now we are in a position to prove Theorem C.

Proof of Theorem C. For $2i-1 \leq 2p-3$ we have

$$\begin{aligned}\pi_{2i-1}F^n\mathrm{TC}(\mathbb{W}_n(k)) &\cong \pi_{2i-1}F^{[n,pn]}\mathrm{TC}(\mathbb{W}_n(k)) \\ &\cong \pi_{2i-1}F^{[n,pn]}\Sigma THH(\mathbb{W}_n(k))_{hS^1} \cong \pi_{2i-1}F^n\Sigma THH(\mathbb{W}_n(k))_{hS^1}.\end{aligned}$$

We then have an exact sequence

$$\begin{aligned}(6.4)\quad 0 \to \pi_{2i-1}F^n\mathrm{TC}(\mathbb{W}_n(k)) &\to \mathrm{TC}_{2i-1}(\mathbb{W}_n(k),(p)) \\ &\to \pi_{2i-1}F^{[1,n-1]}\mathrm{TC}(\mathbb{W}_n(k)) \xrightarrow{\partial} \pi_{2i-2}F^n\mathrm{TC}(\mathbb{W}_n(k)).\end{aligned}$$

Here we can think of ∂ as representing all the differentials in the spectral sequence converging to $\mathrm{TC}_*(\mathbb{W}_n(k))$ crossing filtration n. If $i < p-1$ or $i = p-1$ and $n > p$ then ∂ is surjective, in the case $i = p-1$ and $n \leq p$ the cokernel is $\mathrm{coker}(F-1)$ generated by the image of $\partial\lambda_1 \in \pi_{2p-2}F^n\mathrm{TC}(\mathbb{W}(k))$.

Let $\xi_{2i-1}(1)$ denote the a lift of the generator in filtration 1 of the spectral sequence computing $\mathrm{TF}_{2i-1}(\mathbb{W}_n(k),(p))$. By comparing to the spectral sequence converging to $\pi_*\Sigma THH(\mathbb{W}_n(k))_{hS^1}$ we see that the classes in $\ker\partial$ represent, up to higher filtration, multiples of $\xi_{2i-1}(1)$.

Next, by comparing to $\mathrm{TC}(\mathbb{W}(k))$ we know that for $2i-1 \leq 2p-5$ we have maximally nontrivial extensions in $\ker\partial$. In degree $2p-3$ we find that $\ker\partial$ is a direct sum of k and a maximally nontrivial extension. Similarly, by comparing $F^n\mathrm{TC}(\mathbb{W}_n(k))$ to $F^n\Sigma THH(\mathbb{W}_n(k))_{hS^1}$ we find that all the extensions in $\pi_{2i-1}F^n\mathrm{TC}(\mathbb{W}_n(k))$ are nontrivial.

Now, to prove that we have a maximally nontrivial extension we use Proposition 6.1 and 6.2. We know there are maximally nontrivial extensions in the corresponding spectral sequence converging to $\pi_*\mathrm{TC}(\mathbb{W}(k),(p))$, and a counting argument in the spectral sequence converging to $\pi_*\mathrm{TC}(\mathbb{W}_n(k),(p))$ shows that there is at least one surviving class in filtration $\geq n$ in the range where $\pi_{2i-1}\mathrm{TC}(\mathbb{W}_n(k))$ is isomorphic to $\pi_{2i-1}\mathrm{TC}(\mathbb{W}(k))$ in the sense of the above propositions.

This finishes the proof, because we have shown that in Equation 6.4 the group $\pi_{2i-1}F^n\mathrm{TC}(\mathbb{W}_n(k))$ has the required extensions and similarly for $\ker(\partial)$ except for the case $2i-1 = 2p-3$, and that the extension is maximally nontrivial. $\square$

We finish by recording what this means when k is finite, in which case we find the following.

Corollary 6.5. *Suppose $k = \mathbb{F}_{p^s}$ is a finite field with p^s elements. Then*

$$\begin{aligned}&K_{2i-1}(\mathbb{W}_n(k),(p)) \\ &\quad\cong \begin{cases}(\mathbb{Z}/p^{(n-1)i})^s & \text{for } 1 \leq 2i-1 \leq 2p-5 \\ \mathbb{Z}/p \oplus \mathbb{Z}/p^{(n-1)(p-1)-1} \oplus (\mathbb{Z}/p^{(n-1)(p-1)})^{s-1} & \text{for } 2i-1 = 2p-3\end{cases}\end{aligned}$$

and

$$K_{2i}(\mathbb{W}_n(k),(p)) \cong \begin{cases}0 & \text{for } 2 \leq 2i \leq 2p-4 \\ \mathbb{Z}/p & \text{for } 2i = 2p-2\end{cases}$$

In particular,

$$K_{2i-1}(\mathbb{Z}/p^n,(p)) \cong \begin{cases}\mathbb{Z}/p^{(n-1)i} & \text{for } 1 \leq 2i-1 \leq 2p-5 \\ \mathbb{Z}/p \oplus \mathbb{Z}/p^{(n-1)(p-1)-1} & \text{for } 2i-1 = 2p-3\end{cases}$$

and

$$K_{2i}(\mathbb{Z}/p^n, (p)) \cong \begin{cases} 0 & \text{for } 2 \le 2i \le 2p-4 \\ \mathbb{Z}/p & \text{for } 2i = 2p-2 \end{cases}$$

References

[1] Vigleik Angeltveit, Andrew Blumberg, Teena Gerhardt, Michael A. Hill, and Tyler Lawson. THH and the norm. *In preparation.*

[2] Vigleik Angeltveit and Teena Gerhardt. $RO(S^1)$-graded TR-groups of $\mathbb{F}_p$, $\mathbb{Z}$ and ℓ. *J. Pure Appl. Algebra*, 215(6):1405–1419, 2011.

[3] Vigleik Angeltveit, Michael A. Hill, and Tyler Lawson. Topological Hochschild homology of ℓ and ko. *Amer. J. Math.*, 132(2):297–330, 2010.

[4] J. Michael Boardman. Conditionally convergent spectral sequences. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 49–84. Amer. Math. Soc., Providence, RI, 1999.

[5] M. Bökstedt, W. C. Hsiang, and I. Madsen. The cyclotomic trace and algebraic K-theory of spaces. *Invent. Math.*, 111(3):465–539, 1993.

[6] M. Bökstedt and I. Madsen. Topological cyclic homology of the integers. *Astérisque*, (226):7–8, 57–143, 1994. K-theory (Strasbourg, 1992).

[7] M. Bökstedt and I. Madsen. Algebraic K-theory of local number fields: the unramified case. In *Prospects in topology (Princeton, NJ, 1994)*, volume 138 of *Ann. of Math. Stud.*, pages 28–57. Princeton Univ. Press, Princeton, NJ, 1995.

[8] Marcel Bökstedt. Topological hochschild homology. *Unpublished.*

[9] Marcel Bökstedt. The topological hochschild homology of $\mathbb{Z}$ and $\mathbb{Z}/p$. *Unpublished.*

[10] M. Brun. Topological Hochschild homology of $\mathbf{Z}/p^n$. *J. Pure Appl. Algebra*, 148(1):29–76, 2000.

[11] Morten Brun. Filtered topological cyclic homology and relative K-theory of nilpotent ideals. *Algebr. Geom. Topol.*, 1:201–230 (electronic), 2001.

[12] Morten Brun, Zbigniew Fiedorowicz, and Rainer M. Vogt. On the multiplicative structure of topological Hochschild homology. *Algebr. Geom. Topol.*, 7:1633–1650, 2007.

[13] B. Dundas, Goodwillie T., and R. McCarthy. *The local structure of algebraic K-theory.* In preparation.

[14] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory.* American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

[15] Leonard Evens and Eric M. Friedlander. On $K_*(\mathbf{Z}/p^2\mathbf{Z})$ and related homology groups. *Trans. Amer. Math. Soc.*, 270(1):1–46, 1982.

[16] Thomas Geisser. On K_3 of Witt vectors of length two over finite fields. *K-Theory*, 12(3):193–226, 1997.

[17] Teena Gerhardt. The RS^1-graded equivariant homotopy of $THH(\mathbb{F}_p)$. *Algebraic and Geometric Topology*, 8(4):1961–1987, 2008.

[18] J. P. C. Greenlees and J. P. May. Generalized Tate cohomology. *Mem. Amer. Math. Soc.*, 113(543):viii+178, 1995.

[19] Lars Hesselholt. The tower of K-theory of truncated polynomial algebras. *J. Topol.*, 1(1):87–114, 2008.

[20] Lars Hesselholt and Ib Madsen. Cyclic polytopes and the K-theory of truncated polynomial algebras. *Invent. Math.*, 130(1):73–97, 1997.

[21] Lars Hesselholt and Ib Madsen. On the K-theory of finite algebras over Witt vectors of perfect fields. *Topology*, 36(1):29–101, 1997.

[22] Lars Hesselholt and Ib Madsen. On the K-theory of local fields. *Ann. of Math. (2)*, 158(1):1–113, 2003.

[23] Mark Hovey, Brooke Shipley, and Jeff Smith. Symmetric spectra. *J. Amer. Math. Soc.*, 13(1):149–208, 2000.

[24] Ch. Kratzer. λ-structure en K-théorie algébrique. *Comment. Math. Helv.*, 55(2):233–254, 1980.

[25] J. P. May. *Equivariant homotopy and cohomology theory*, volume 91 of *CBMS Regional Conference Series in Mathematics.* Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996. With contributions by M. Cole, G. Comezaña, S. Costenoble,

A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner.

[26] J. Peter May. A general algebraic approach to Steenrod operations. In *The Steenrod Algebra and its Applications (Proc. Conf. to Celebrate N. E. Steenrod's Sixtieth Birthday, Battelle Memorial Inst., Columbus, Ohio, 1970)*, Lecture Notes in Mathematics, Vol. 168, pages 153–231. Springer, Berlin, 1970.

[27] Randy McCarthy. Relative algebraic K-theory and topological cyclic homology. *Acta Math.*, 179(2):197–222, 1997.

[28] J. E. McClure and R. E. Staffeldt. On the topological Hochschild homology of bu. I. *Amer. J. Math.*, 115(1):1–45, 1993.

[29] John Rognes. Trace maps from the algebraic K-theory of the integers (after Marcel Bökstedt). *J. Pure Appl. Algebra*, 125(1-3):277–286, 1998.

[30] John Rognes. Algebraic K-theory of the two-adic integers. *J. Pure Appl. Algebra*, 134(3):287–326, 1999.

[31] John Rognes. The product on topological Hochschild homology of the integers with mod 4 coefficients. *J. Pure Appl. Algebra*, 134(3):211–218, 1999.

[32] John Rognes. Topological cyclic homology of the integers at two. *J. Pure Appl. Algebra*, 134(3):219–286, 1999.

[33] Stavros Tsalidis. Topological Hochschild homology and the homotopy descent problem. *Topology*, 37(4):913–934, 1998.

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